# Binomial Determinants for Tiling Problems Yield to the Holonomic Ansatz

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$$D^{\mu}_{s,t}(n) := \det_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n}} \left( \binom{\mu + i + j + s + t - 4}{j + t - 1} + \delta_{i+s,j+t} \right),$$

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**History:** $D^{\mu}_{0,0}(n)$ was introduced in the work of Andrews in 1979–1980 in the context of descending plane partitions:

Inventiones math. 53, 193–225 (1979)

*Inventiones mathematicae*
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**Plane Partitions (III): The Weak Macdonald Conjecture**

George E. Andrews*
The Pennsylvania State University, University Park, Pennsylvania 16802, U.S.A.

Dedicated to the memory of Alfred Young and F.J.W. Whipple

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**Example:** $D^{\mu}_{4,6}(5)$ is the determinant of the matrix

$$\begin{pmatrix}
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**Example:** $D^{\mu-1}_{5,6}(5)$ is the determinant of the matrix

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**Definition:** For $n \in \mathbb{N}$, for $s, t \in \mathbb{Z}$, and for $\mu$ an indeterminate, define the following $(n \times n)$-determinants:

$$D_{s,t}^{\mu}(n) := \det_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n}} \left( \binom{\mu + i + j + s + t - 4}{j + t - 1} + \delta_{i+s,j+t} \right),$$

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**Example:** $E_{6,5}^{\mu-1}(5)$ is the determinant of the matrix

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# Evaluating Determinants: Laplace Expansion



Consider the matrix for $s = 2, t = 1, n = 2$:

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Specializing $\mu = 2$, we can compute the determinant using algebra

$$E^2_{2,1}(2) = \det \begin{pmatrix} \binom{3}{1} & \binom{4}{2} - 1 \\ \binom{4}{1} & \binom{5}{2} \end{pmatrix} = \binom{3}{1}\binom{5}{2} - \left(\binom{4}{2} - 1\right)\binom{4}{1}$$

$$= \underbrace{\left(\binom{3}{1}\binom{5}{2} - \binom{4}{2}\binom{4}{1}\right)}_{=6} + \underbrace{1\binom{4}{1}}_{=4}$$

$$= 10.$$

# Lindström-Gessel-Viennot Lemma



Let $G$ be a directed acyclic graph and consider base vertices $A = \{a_1, \ldots, a_n\}$ and destination vertices $B = \{b_1, \ldots, b_n\}$.

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$$e(a, b) = \sum_{P: a \to b} \omega(P) \qquad \text{and}$$

$$M = \begin{pmatrix} e(a_1, b_1) & e(a_1, b_2) & \cdots & e(a_1, b_n) \\ e(a_2, b_1) & e(a_2, b_2) & \cdots & e(a_2, b_n) \\ \vdots & \vdots & \ddots & \vdots \\ e(a_n, b_1) & e(a_n, b_2) & \cdots & e(a_n, b_n) \end{pmatrix}.$$

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Then the determinant of $M$ is the signed sum over all $n$-tuples $P = (P_1, \ldots, P_n)$ of non-intersecting paths from $A$ to $B$:

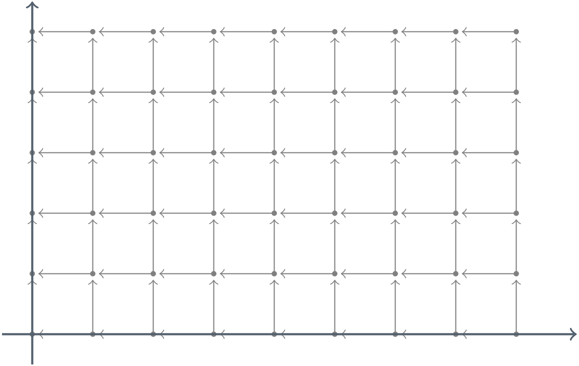
$$\det(M) = \sum_{(P_1, \ldots, P_n):\ A \to B} \mathrm{sign}(\sigma(P)) \prod_{i=1}^{n} \omega(P_i).$$

where $\sigma$ denotes a permutation that is applied to $B$.

In our context, the determinant without the Kronecker delta

$$\det_{1\leqslant i,j\leqslant n} \binom{\mu+i+j+s+t-4}{j+t-1}$$

counts $n$-tuples of non-intersecting paths in the lattice $\mathbb{N}^2$:



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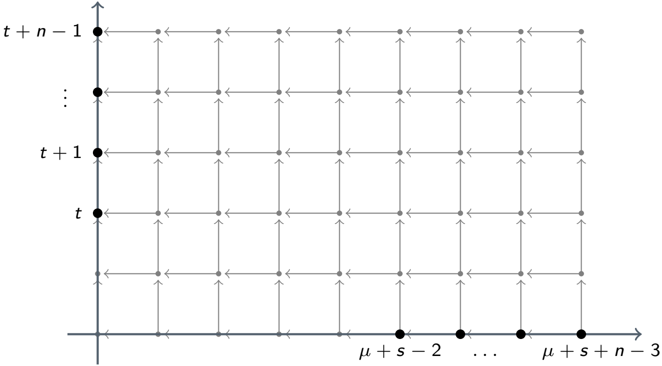

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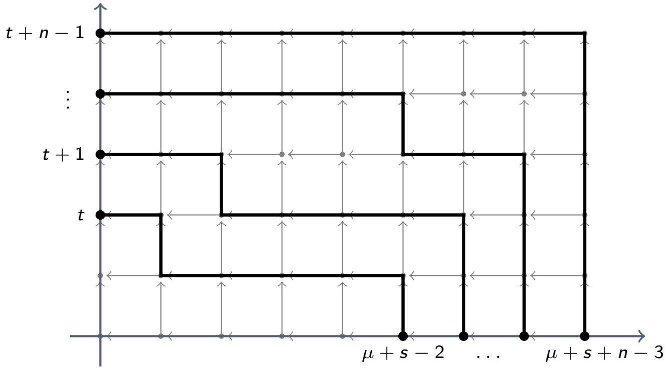
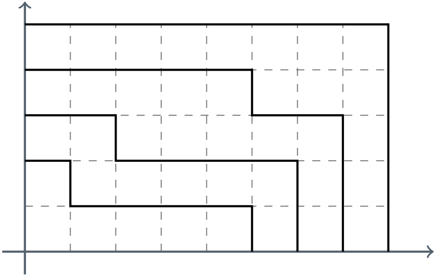
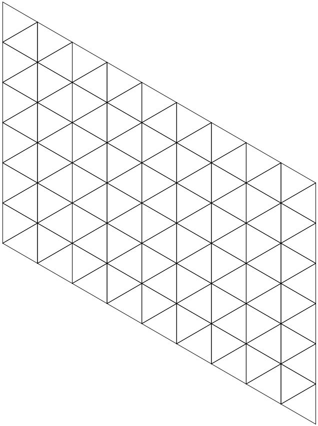
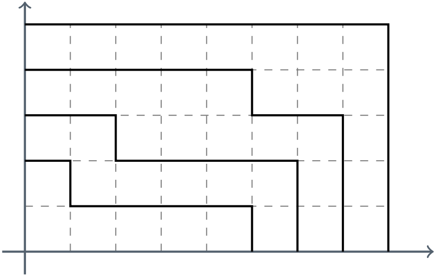


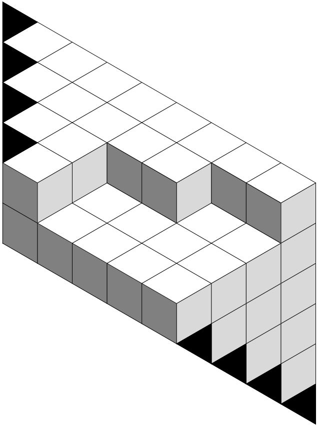
# Lattice Paths $\longleftrightarrow$ Rhombus Tilings











Back to our problem:

$$E^2_{2,1}(2) = \det \begin{pmatrix} \binom{3}{1} & \binom{4}{2} - 1 \\ \binom{4}{1} & \binom{5}{2} \end{pmatrix} = 10$$

For $1 \leqslant i, j \leqslant n$:

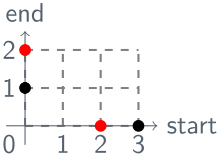
$$\binom{\mu + s + t + i + j - 4}{j + t - 1} \to \begin{cases} \text{start:} & (\mu + s + i - 3, 0), \\ \text{end:} & (0, j + t - 1). \end{cases}$$

# Evaluating Determinants: Rhombus Tilings



For $E^{2}_{2,1}(2)$, we have that $n = 2$ implying $i, j \in \{1, 2\}$ and

$$\binom{2+2+1+i+j-4}{j+1-1} \to \begin{cases} \text{start:} & (2+2+i-3, 0) \\ \text{end:} & (0, j+1-1) \end{cases} \to \begin{cases} \text{start:} & (i+1, 0), \\ \text{end:} & (0, j). \end{cases}$$

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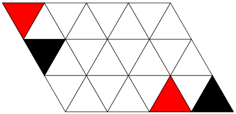

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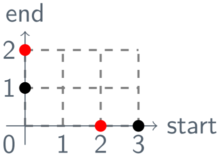


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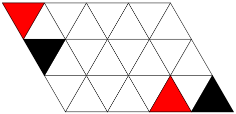

There are six ways to realize these 2-tuples and tile this region:

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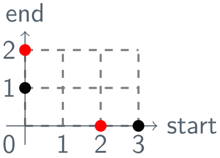


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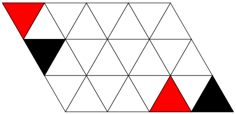

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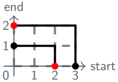


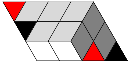

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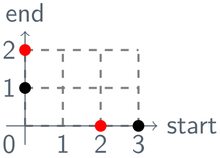


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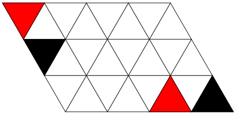

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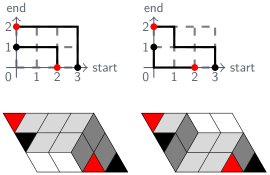

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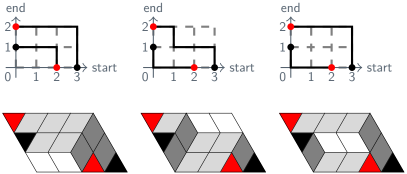

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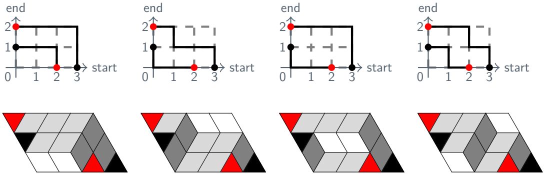
$$\binom{2+2+1+i+j-4}{j+1-1} \to \begin{cases} \text{start:} & (2+2+i-3, 0) \\ \text{end:} & (0, j+1-1) \end{cases} \to \begin{cases} \text{start:} & (i+1, 0), \\ \text{end:} & (0, j). \end{cases}$$


There are six ways to realize these 2-tuples and tile this region:



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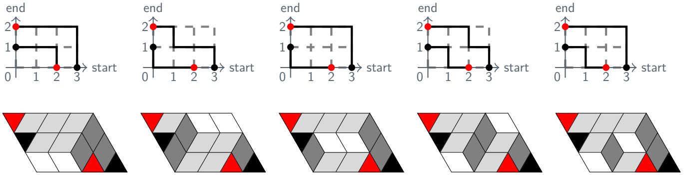

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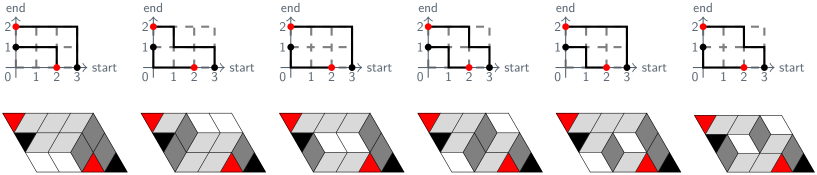
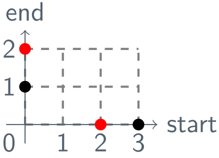


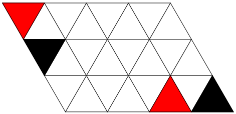
# Evaluating Determinants: Rhombus Tilings



The minor(s) associated to the Kronecker delta(s) corresponds to the *removal* of starting and ending points:




$\leftrightarrow$



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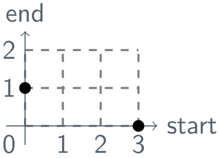


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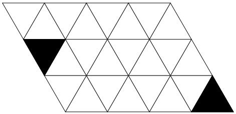

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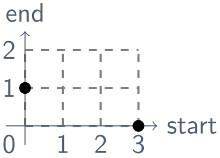
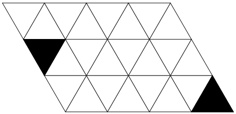
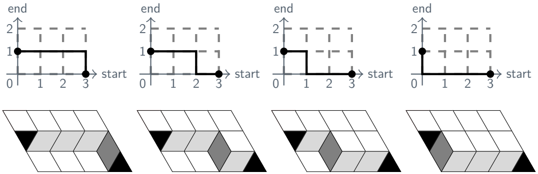


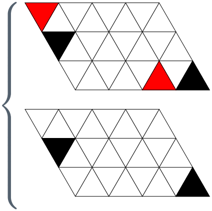

↔



There are four ways to realize the 1-tuples and tile this region:



We can reframe this into one tiling problem, namely, to count the number of cyclically symmetric tilings of one holey hexagonal region:



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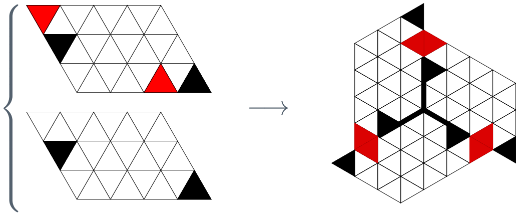

# Evaluating Determinants: Rhombus Tilings



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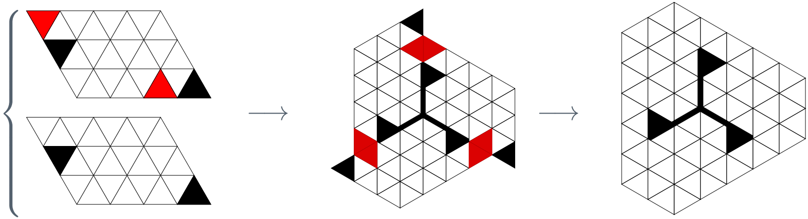

# Evaluating Determinants: Rhombus Tilings

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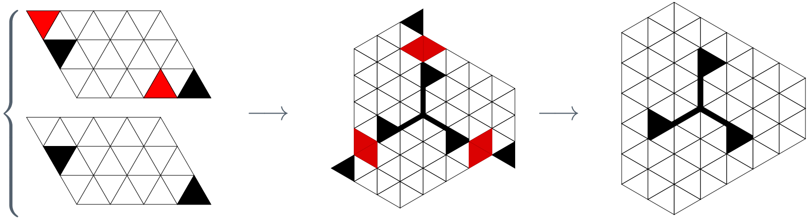
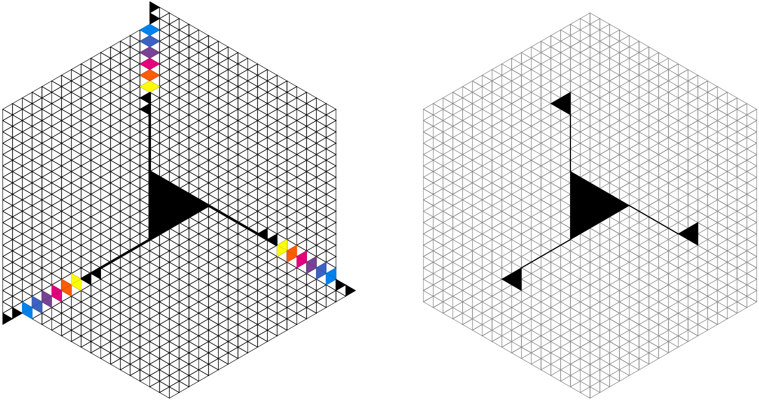


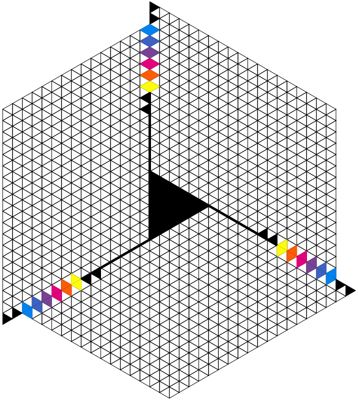
$$E_{2,1}^{2}(2) = 10$$

# Evaluating Determinants: Rhombus Tilings



The region associated to the determinant $D^{8}_{5,7}(8)$ and an illustration of one cyclically symmetric tiling of this region.

# Evaluating Determinants: Rhombus Tilings



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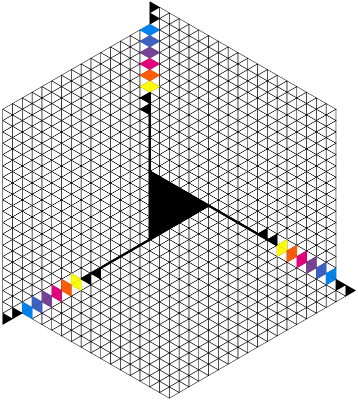

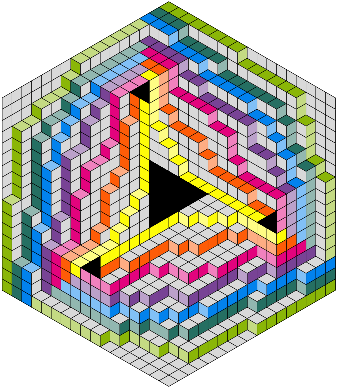

The region associated to the determinant $D^{8}_{5,7}(8)$ and an illustration of one cyclically symmetric tiling of this region.

# A Combinatorial Proof



**Lemma:** For $n, s \in \mathbb{Z}$ such that $n \geqslant s \geqslant 1$ and $n > 1$,

$$E^{\mu}_{s,0}(n) = D^{\mu+3}_{s-1,0}(n-1),$$
$$D^{\mu}_{s,0}(n) = E^{\mu+3}_{s-1,0}(n-1).$$

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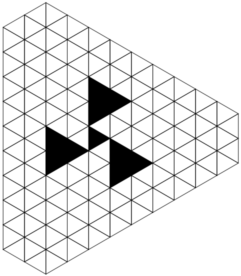
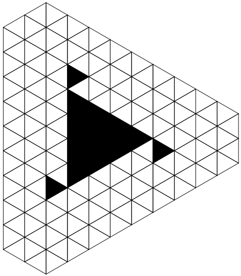
**Proof (by example):**

$D^3_{2,0}(4)$



$E^6_{1,0}(3)$



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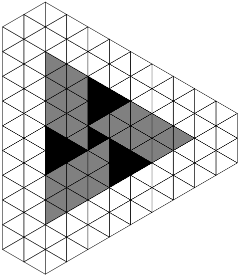

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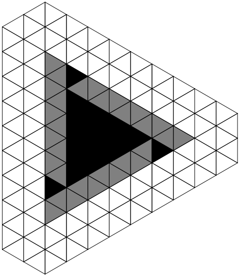
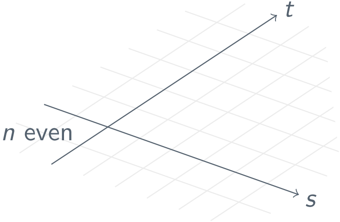


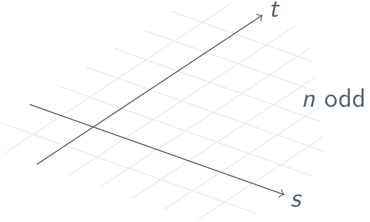
# The Big Picture

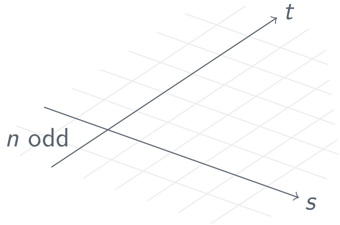


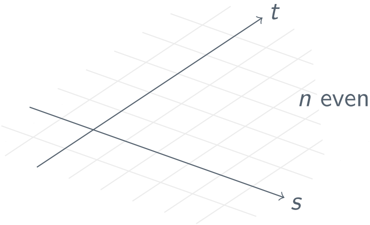
$E^{\mu}_{s,t}(n)$ Family

$D^{\mu}_{s,t}(n)$ Family



$E^{\mu}_{s,t}(n)$ Family

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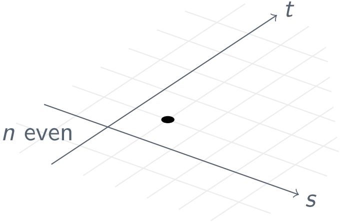


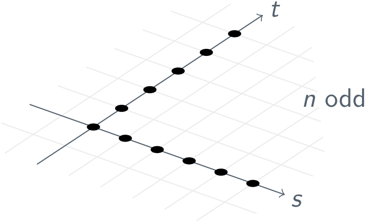


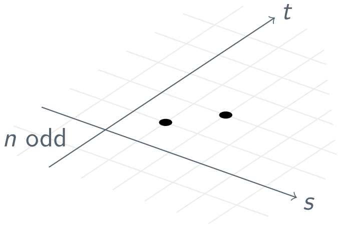


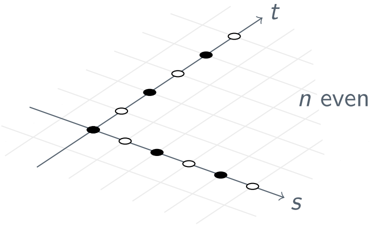

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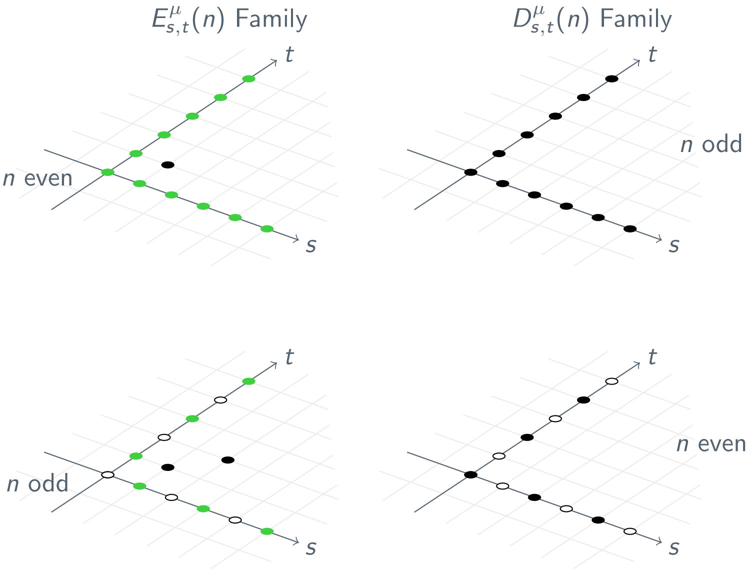

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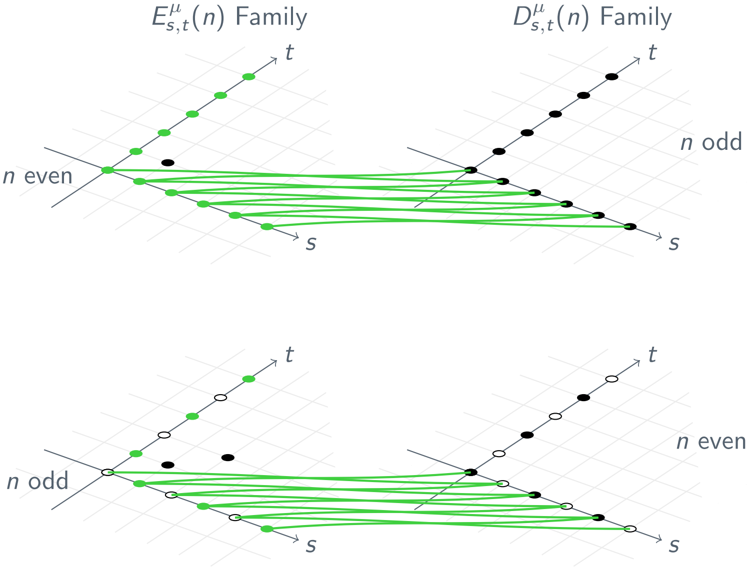

# Switching

## Lemma

*Let $A^{\mu}_{s,t}(n)$ be either $D^{\mu}_{s,t}(n)$ or $E^{\mu}_{s,t}(n)$. For real numbers $s, t \notin \{-1, -2, \ldots\}$ with $t - s \in \mathbb{N}$ and $n \in \mathbb{Z}^{+}$,*

$$A^{\mu}_{s,t}(n) = \prod_{i=0}^{t-s-1} \frac{\left(\mu + s + i - 1\right)_n}{\left(i + s + 1\right)_n} \cdot A^{\mu}_{t,s}(n).$$

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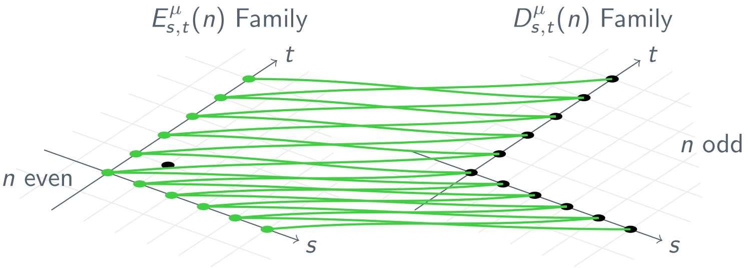
**The Pochhammer Symbol:**

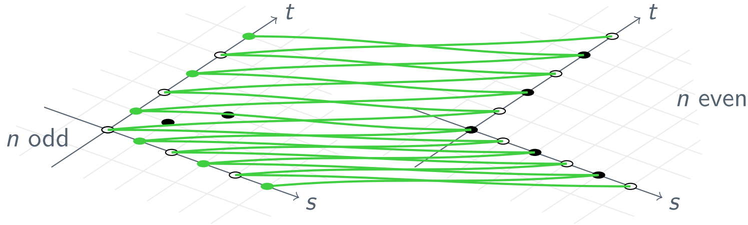
For an indeterminate $x$, and $y \in \mathbb{Z}$ :

$$(x)_y := \begin{cases} x(x+1)\cdots(x+y-1), & y > 0, \\ 1, & y = 0, \\ \frac{1}{(x+y)_{-y}}, & y < 0. \end{cases}$$

# The Big Picture





# Determinant Closed Forms



## Conjecture 37 (Krattenthaler and Lascoux, 2005)

Let $\mu$ be an indeterminate and $m, r \in \mathbb{Z}$. If $m \geqslant r \geqslant 1$, then

$$E^{\mu}_{1,2r-1}(2m-1) = (-1)^{m-r} \cdot 2^{m^2-2mr+3m+r^2-2r} \cdot \prod_{i=0}^{m-1} \frac{i!\,(i+1)!}{(2i)!\,(2i+2)!}$$

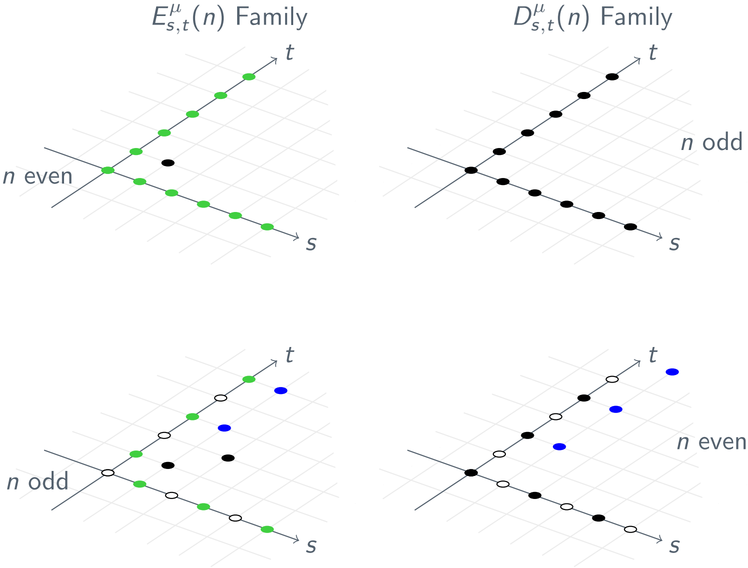
$$\times \prod_{i=0}^{2r-3} i! \cdot \prod_{i=0}^{r-2} \frac{((2m-2i-3)!)^2}{((m-i-2)!)^2\,(2m+2i-1)!\,(2m+2i+1)!}$$

$$\times (\mu-1) \cdot \left(\frac{\mu}{2}+r-\frac{1}{2}\right)_{m-r} \cdot \prod_{i=1}^{2r-2} (\mu+i-1)_{2m+2r-2i-1}$$

$$\times \prod_{i=0}^{\left\lfloor \frac{m-r-1}{2} \right\rfloor} \left(\frac{\mu}{2}+3i+3r-\frac{1}{2}\right)^2_{m-r-2i-1} \cdot \left(-\frac{\mu}{2}-3m+3i+3\right)^2_{m-r-2i}.$$

# The Big Picture

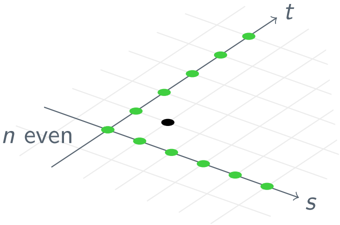


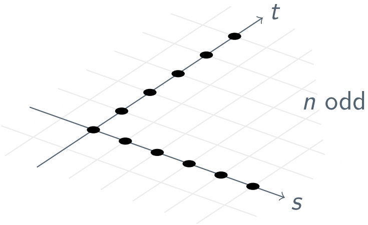


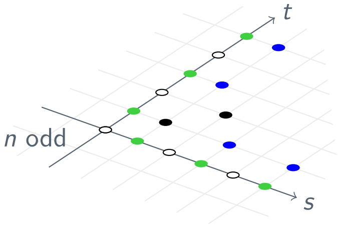
# The Big Picture

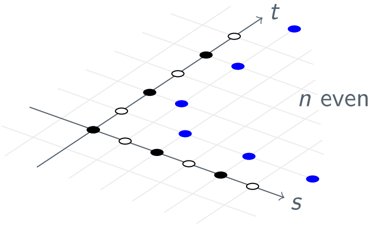
$E_{s,t}^{\mu}(n)$ Family

$D_{s,t}^{\mu}(n)$ Family



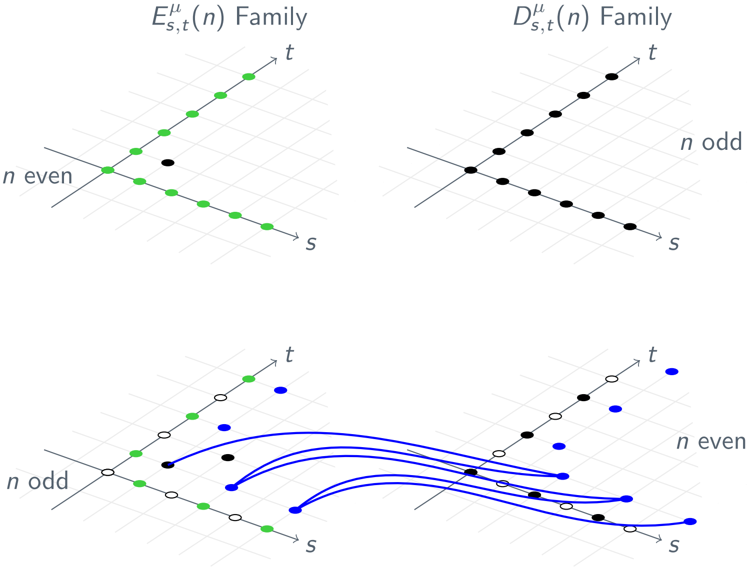



# The Big Picture





# Towards the Proof of the Conjecture



## Lemma

*For $\mu$ indeterminate, $m, r \in \mathbb{Z}$ and $m \geqslant r \geqslant 1$,*

$$\frac{D^{\mu}_{2r,1}(2m)}{E^{\mu+3}_{2r-1,1}(2m-1)} = \frac{(m+r-1)(\mu-1)(\mu+2m+1)(\mu+2r)}{2m(2r-1)(\mu+2)(\mu+2m+2r-1)},$$

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## Theorem

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$$E^{\mu}_{2r-1,1}(2m-1) = \frac{(-1)^{m-r}(\mu-1)\left(\mu+2r-1\right)_{2m-2}}{(2r-2)!\left(m+r-1\right)_{m-r+1}\left(\frac{\mu}{2}+r\right)_{m-r}} \times \prod_{i=1}^{m-r} \frac{\left(\mu+2i+6r-5\right)^2_{i-1}\left(\frac{\mu}{2}+2i+3r-2\right)^2_{i}}{\left(i\right)^2_{i}\left(\frac{\mu}{2}+i+3r-2\right)^2_{i-1}}.$$

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**Corollary:** Apply switching lemma to obtain $E^{\mu}_{1,2r-1}(2m-1)$.

# Towards the Proof of the Conjecture



## Lemma

*For $\mu$ indeterminate, $n, s \in \mathbb{Z}$ and $n \geqslant s \geqslant 1$,*

$$\frac{A^{\mu}_{s,1}(n)}{B^{\mu+3}_{s-1,1}(n-1)} = \frac{(n+s-2)(\mu-1)(\mu+n+1)(\mu+s)}{2n(s-1)(\mu+2)(\mu+n+s-1)},$$

*where $(A, B, s, n)$ is $(D, E, 2r, 2m)$ or $(E, D, 2r+1, 2m+1)$.*

## Theorem

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# Matrix Transformations



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \mathcal{D}^{\mu}_{2,1}(4) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+2}{2}+1 & \binom{\mu+3}{3} & \binom{\mu+4}{4} \\ \binom{\mu+2}{1} & \binom{\mu+3}{2} & \binom{\mu+4}{3}+1 & \binom{\mu+5}{4} \\ \binom{\mu+3}{1} & \binom{\mu+4}{2} & \binom{\mu+5}{3} & \binom{\mu+6}{4}+1 \\ \binom{\mu+4}{1} & \binom{\mu+5}{2} & \binom{\mu+6}{3} & \binom{\mu+7}{4} \end{pmatrix}$$

# Matrix Transformations



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}^{\mu}_{2,1}(4) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+2}{2} + 1 & \binom{\mu+3}{3} & \binom{\mu+4}{4} \\ \binom{\mu+2}{1} - \binom{\mu+1}{1} & \binom{\mu+3}{2} - \binom{\mu+2}{2} - 1 & \binom{\mu+4}{3} - \binom{\mu+3}{3} + 1 & \binom{\mu+5}{4} - \binom{\mu+4}{4} \\ \binom{\mu+3}{1} - \binom{\mu+2}{1} & \binom{\mu+4}{2} - \binom{\mu+3}{2} & \binom{\mu+5}{3} - \binom{\mu+4}{3} - 1 & \binom{\mu+6}{4} - \binom{\mu+5}{4} + 1 \\ \binom{\mu+4}{1} - \binom{\mu+3}{1} & \binom{\mu+5}{2} - \binom{\mu+4}{2} & \binom{\mu+6}{3} - \binom{\mu+5}{3} & \binom{\mu+7}{4} - \binom{\mu+6}{4} - 1 \end{pmatrix}$$

# Matrix Transformations



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$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+2}{2}+1 & \binom{\mu+3}{3} & \binom{\mu+4}{4} \\ \binom{\mu+1}{0} & \binom{\mu+2}{1}-1 & \binom{\mu+3}{2}+1 & \binom{\mu+4}{3} \\ \binom{\mu+2}{0} & \binom{\mu+3}{1} & \binom{\mu+4}{2}-1 & \binom{\mu+5}{3}+1 \\ \binom{\mu+3}{0} & \binom{\mu+4}{1} & \binom{\mu+5}{2} & \binom{\mu+6}{3}-1 \end{pmatrix}$$

# Matrix Transformations



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}^{\mu}_{2,1}(4) \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+2}{2} + \binom{\mu+1}{1} + 1 & \binom{\mu+3}{3} & \binom{\mu+4}{4} \\ \binom{\mu+1}{0} & \binom{\mu+2}{1} + \binom{\mu+1}{0} - 1 & \binom{\mu+3}{2} + 1 & \binom{\mu+4}{3} \\ \binom{\mu+2}{0} & \binom{\mu+3}{1} + \binom{\mu+2}{0} & \binom{\mu+4}{2} - 1 & \binom{\mu+5}{3} + 1 \\ \binom{\mu+3}{0} & \binom{\mu+4}{1} + \binom{\mu+3}{0} & \binom{\mu+5}{2} & \binom{\mu+6}{3} - 1 \end{pmatrix}$$

# Matrix Transformations



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}^{\mu}_{2,1}(4) \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+2}{2} + \binom{\mu+2}{1} & \binom{\mu+3}{3} & \binom{\mu+4}{4} \\ \binom{\mu+1}{0} & \binom{\mu+2}{1} + \binom{\mu+2}{0} - 1 & \binom{\mu+3}{2} + 1 & \binom{\mu+4}{3} \\ \binom{\mu+2}{0} & \binom{\mu+3}{1} + \binom{\mu+3}{0} & \binom{\mu+4}{2} - 1 & \binom{\mu+5}{3} + 1 \\ \binom{\mu+3}{0} & \binom{\mu+4}{1} + \binom{\mu+4}{0} & \binom{\mu+5}{2} & \binom{\mu+6}{3} - 1 \end{pmatrix}$$

# Matrix Transformations



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}_{2,1}^{\mu}(4) \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+3}{2} & \binom{\mu+3}{3} & \binom{\mu+4}{4} \\ \binom{\mu+1}{0} & \binom{\mu+3}{1} - 1 & \binom{\mu+3}{2} + 1 & \binom{\mu+4}{3} \\ \binom{\mu+2}{0} & \binom{\mu+4}{1} & \binom{\mu+4}{2} - 1 & \binom{\mu+5}{3} + 1 \\ \binom{\mu+3}{0} & \binom{\mu+5}{1} & \binom{\mu+5}{2} & \binom{\mu+6}{3} - 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}_{2,1}^{\mu}(4) \cdot \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+3}{2} & \binom{\mu+3}{3}+\binom{\mu+3}{2} & \binom{\mu+4}{4} \\ \binom{\mu+1}{0} & \binom{\mu+3}{1}-1 & \binom{\mu+3}{2}+\binom{\mu+3}{1} & \binom{\mu+4}{3} \\ \binom{\mu+2}{0} & \binom{\mu+4}{1} & \binom{\mu+4}{2}+\binom{\mu+4}{1}-1 & \binom{\mu+5}{3}+1 \\ \binom{\mu+3}{0} & \binom{\mu+5}{1} & \binom{\mu+5}{2}+\binom{\mu+5}{1} & \binom{\mu+6}{3}-1 \end{pmatrix}$$

# Matrix Transformations



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}_{2,1}^{\mu}(4) \cdot \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

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# Matrix Transformations



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}^{\mu}_{2,1}(4) \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+3}{2} & \binom{\mu+4}{3} & \binom{\mu+4}{4}+\binom{\mu+4}{3} \\ \binom{\mu+1}{0} & \binom{\mu+3}{1}-1 & \binom{\mu+4}{2} & \binom{\mu+4}{3}+\binom{\mu+4}{2} \\ \binom{\mu+2}{0} & \binom{\mu+4}{1} & \binom{\mu+5}{2}-1 & \binom{\mu+5}{3}+\binom{\mu+5}{2} \\ \binom{\mu+3}{0} & \binom{\mu+5}{1} & \binom{\mu+6}{2} & \binom{\mu+6}{3}+\binom{\mu+6}{2}-1 \end{pmatrix}$$

# Matrix Transformations



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{D}^{\mu}_{2,1}(4) \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+3}{2} & \binom{\mu+4}{3} & \binom{\mu+5}{4} \\ \binom{\mu+1}{0} & \binom{\mu+3}{1} - 1 & \binom{\mu+4}{2} & \binom{\mu+5}{3} \\ \binom{\mu+2}{0} & \binom{\mu+4}{1} & \binom{\mu+5}{2} - 1 & \binom{\mu+6}{3} \\ \binom{\mu+3}{0} & \binom{\mu+5}{1} & \binom{\mu+6}{2} & \binom{\mu+7}{3} - 1 \end{pmatrix}$$

$$\mathcal{L} \cdot \mathcal{D}^{\mu}_{2,1}(4) \cdot \mathcal{R} = \left(\begin{array}{c|ccc} * & * & * & * \\ \hline 1 & & & \\ 1 & & \mathcal{E}^{\mu+3}_{1,1}(3) & \\ 1 & & & \end{array}\right)$$

# Matrix Transformations



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{E}^{\mu}_{2,1}(4) \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+2}{2} - 1 & \binom{\mu+3}{3} & \binom{\mu+4}{4} \\ \binom{\mu+2}{1} & \binom{\mu+3}{2} & \binom{\mu+4}{3} - 1 & \binom{\mu+5}{4} \\ \binom{\mu+3}{1} & \binom{\mu+4}{2} & \binom{\mu+5}{3} & \binom{\mu+6}{4} - 1 \\ \binom{\mu+4}{1} & \binom{\mu+5}{2} & \binom{\mu+6}{3} & \binom{\mu+7}{4} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \mathcal{E}_{2,1}^{\mu}(4) \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \binom{\mu+1}{1} & \binom{\mu+3}{2} & \binom{\mu+4}{3} & \binom{\mu+5}{4} \\ \binom{\mu+1}{0} & \binom{\mu+3}{1}+1 & \binom{\mu+4}{2} & \binom{\mu+5}{3} \\ \binom{\mu+3}{1} & \binom{\mu+4}{1} & \binom{\mu+5}{2}+1 & \binom{\mu+6}{3} \\ \binom{\mu+3}{0} & \binom{\mu+5}{2} & \binom{\mu+6}{2} & \binom{\mu+7}{3}+1 \end{pmatrix}$$

$$\mathcal{L} \cdot \mathcal{E}_{2,1}^{\mu}(4) \cdot \mathcal{R} = \left(\begin{array}{c|ccc} * & * & * & * \\ \hline 1 & & & \\ 1 & & \mathcal{D}_{1,1}^{\mu+3}(3) & \\ 1 & & & \end{array}\right)$$

# Proof of Lemma via the Holonomic Ansatz



For $\mu$ indeterminate, $n, s \in \mathbb{Z}$ and $n \geqslant s \geqslant 1$,

$$\frac{A^{\mu}_{s,1}(n)}{B^{\mu+3}_{s-1,1}(n-1)} = \underbrace{\frac{(n+s-2)(\mu-1)(\mu+n+1)(\mu+s)}{2n(s-1)(\mu+2)(\mu+n+s-1)}}_{=:R^{\mu}_{s,1}(n)},$$

where $(A, B, s, n)$ is $(D, E, 2r, 2m)$ or $(E, D, 2r+1, 2m+1)$.

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where $(A, B, s, n)$ is $(D, E, 2r, 2m)$ or $(E, D, 2r+1, 2m+1)$.

**Laplace expansion:**

$$A^{\mu}_{s,1}(n) = \det \left( \begin{array}{c|ccc} \tilde{a}_{1,1} & \tilde{a}_{1,2} & \cdots & \tilde{a}_{1,n} \\ \hline 1 & & & \\ 1 & & \mathcal{B}^{\mu+3}_{s-1,1}(n-1) & \\ 1 & & & \end{array} \right)$$

$$= \tilde{a}_{1,1} \cdot \mathrm{Cof}_{1,1}(n-1) + \ldots + \tilde{a}_{1,n} \cdot \mathrm{Cof}_{1,n}(n-1).$$

# Proof of Lemma via the Holonomic Ansatz



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$$= \tilde{a}_{1,1} \cdot \mathrm{Cof}_{1,1}(n-1) + \ldots + \tilde{a}_{1,n} \cdot \mathrm{Cof}_{1,n}(n-1).$$

With $c_{n,j} := \mathrm{Cof}_{1,j}(n-1)/\mathrm{Cof}_{1,1}(n-1)$, we obtain

$$\frac{A^{\mu}_{s,1}(n)}{B^{\mu+3}_{s-1,1}(n-1)} = \sum_{j=1}^{n} \tilde{a}_{1,j} \cdot c_{n,j}$$

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$$A^{\mu}_{s,1}(n) = \det \begin{pmatrix} \tilde{a}_{1,1} & \tilde{a}_{1,2} \quad \cdots \quad \tilde{a}_{1,n} \\ 1 & \\ 1 & \mathcal{B}^{\mu+3}_{s-1,1}(n-1) \\ 1 & \end{pmatrix}$$

$$= \tilde{a}_{1,1} \cdot \mathrm{Cof}_{1,1}(n-1) + \ldots + \tilde{a}_{1,n} \cdot \mathrm{Cof}_{1,n}(n-1).$$

With $c_{n,j} := \mathrm{Cof}_{1,j}(n-1)/\mathrm{Cof}_{1,1}(n-1)$, we obtain

$$\frac{A^{\mu}_{s,1}(n)}{B^{\mu+3}_{s-1,1}(n-1)} = \sum_{j=1}^{n} \tilde{a}_{1,j} \cdot c_{n,j} \overset{!}{=} R^{\mu}_{s,1}(n).$$

# Proof via the Holonomic Ansatz



**Guess:** $c_{n,j}$ satisfies a holonomic system of recurrence equations.

# Proof via the Holonomic Ansatz



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# Proof via the Holonomic Ansatz



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$$
\begin{aligned}
p^{[1]}_{0,2} \cdot c_{n,j+2} + p^{[1]}_{1,0} \cdot c_{n+1,j} + p^{[1]}_{0,1} \cdot c_{n,j+1} + p^{[1]}_{0,0} \cdot c_{n,j} &= 0 \\
p^{[2]}_{1,1} \cdot c_{n+1,j+1} + p^{[2]}_{1,0} \cdot c_{n+1,j} + p^{[2]}_{0,1} \cdot c_{n,j+1} + p^{[2]}_{0,0} \cdot c_{n,j} &= 0 \\
p^{[3]}_{2,0} \cdot c_{n+2,j} + p^{[3]}_{1,0} \cdot c_{n+1,j} + p^{[3]}_{0,1} \cdot c_{n,j+1} + p^{[3]}_{0,0} \cdot c_{n,j} &= 0
\end{aligned}
$$

# Proof via the Holonomic Ansatz



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$$p^{[2]}_{1,1} \cdot c_{n+1,j+1} + p^{[2]}_{1,0} \cdot c_{n+1,j} + p^{[2]}_{0,1} \cdot c_{n,j+1} + p^{[2]}_{0,0} \cdot c_{n,j} = 0$$
$$p^{[3]}_{2,0} \cdot c_{n+2,j} + p^{[3]}_{1,0} \cdot c_{n+1,j} + p^{[3]}_{0,1} \cdot c_{n,j+1} + p^{[3]}_{0,0} \cdot c_{n,j} = 0$$

$$p^{[3]}_{2,0} = -(j-2n-4)(j-2n-3)(\mu+6n+5)(\mu+6n+7)(\mu+6n+9)(n+r-1)(n+r)(j+\mu+2n+3)(j+\mu+2n+4)(2j^4+3j^3\mu-6j^3n+j^3+j^2\mu^2-12j^2\mu n-3j^2\mu+12j^2n^2-30j^2n-8j^2-4j\mu^2n-2j\mu^2+24j\mu n^2-8j\mu n-6j\mu+72jn^2+12jn-4j+8\mu^2n^2+4\mu^2n+40\mu n^2+20\mu n+48n^2+24n)(\mu+2n+2r)(\mu+2n+2r+1)(\mu+2n+2r+2)(\mu+2n+2r+3)(\mu+4n+2r+1)$$

# Proof via the Holonomic Ansatz



**Prove:** in the case where $(A, B, s, n)$ is $(D, E, 2r, 2m)$

$$\sum_{j=1}^{2m} \binom{\mu + j + 2r - 1}{j} \cdot c_{2m,j} - \sum_{j=1}^{2r-1} c_{2m,j} = R^{\mu}_{2r,1}(2m).$$

# Proof via the Holonomic Ansatz



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- Abandon the original definition $c_{n,j} := \frac{\mathrm{Cof}_{1,j}(n-1)}{\mathrm{Cof}_{1,1}(n-1)}$.
- Use the conjectured holonomic description for $c_{n,j}$ instead.

# Proof via the Holonomic Ansatz

**Prove:** in the case where $(A, B, s, n)$ is $(D, E, 2r, 2m)$

$$c_{2m,1} = 1,$$

$$\sum_{j=1}^{2m} \binom{\mu + i + j + 2r - 3}{j - 1} \cdot c_{2m,j} - c_{2m,i+2r-2} = 0, \qquad (2 \leqslant i \leqslant 2m),$$

$$\sum_{j=1}^{2m} \binom{\mu + j + 2r - 1}{j} \cdot c_{2m,j} - \sum_{j=1}^{2r-1} c_{2m,j} = R^{\mu}_{2r,1}(2m).$$

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- Use the conjectured holonomic description for $c_{n,j}$ instead.
- The first two identities prove $c_{n,j} = \frac{\mathrm{Cof}_{1,j}(n-1)}{\mathrm{Cof}_{1,1}(n-1)}$.

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$$\sum_{j=1}^{2m} \binom{\mu + j + 2r - 1}{j} \cdot c_{2m,j} - \sum_{j=1}^{2r-1} c_{2m,j} = R^{\mu}_{2r,1}(2m).$$

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- Use the conjectured holonomic description for $c_{n,j}$ instead.
- The first two identities prove $c_{n,j} = \frac{\mathrm{Cof}_{1,j}(n-1)}{\mathrm{Cof}_{1,1}(n-1)}$.
- The third identity proves the claimed quotient of determinants.

# Computational Bottlenecks



Prove that for $m \geqslant r \geqslant 1$ and $2 \leqslant i \leqslant 2m$:

$$c_{2m,1} = 1,$$
$$\sum_{j=1}^{2m} \binom{\mu+i+j+2r-3}{j-1} \cdot c_{2m,j} = c_{2m,i+2r-2},$$
$$\sum_{j=1}^{2m} \binom{\mu+j+2r-1}{j} \cdot c_{2m,j} - \sum_{j=1}^{2r-1} c_{2m,j} = R^{\mu}_{2r,1}(2m).$$

We encountered the following computational challenges:

# Computational Bottlenecks



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$$\sum_{j=1}^{2m} \binom{\mu+j+2r-1}{j} \cdot c_{2m,j} - \sum_{j=1}^{2r-1} c_{2m,j} = R^{\mu}_{2r,1}(2m).$$

We encountered the following computational challenges:

- Creative telescoping for the summation in the second identity did not finish.

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$$\sum_{j=1}^{2m} \binom{\mu+j+2r-1}{j} \cdot c_{2m,j} - \sum_{j=1}^{2r-1} c_{2m,j} = R^{\mu}_{2r,1}(2m).$$

We encountered the following computational challenges:

- Creative telescoping for the summation in the second identity did not finish.
- In the third identity, a singularity appeared in the certificate $Q$ at $j = 1$ (for both summations) and we were not able to automatically certify our telescoper.

# Computational Bottlenecks



Prove that for $m \geqslant r \geqslant 1$ and $2 \leqslant i \leqslant 2m+1$:

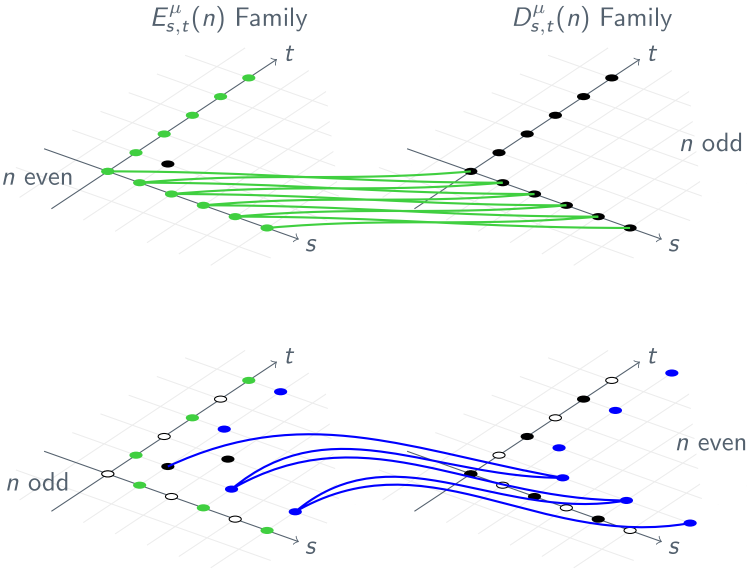
$$\begin{aligned} c_{2m+1,1} &= 1, \\ \sum_{j=1}^{2m+1} \binom{\mu+i+j+2r-2}{j-1} \cdot c_{2m+1,j} + c_{2m+1,i+2r-1} &= 0, \\ \sum_{j=1}^{2m+1} \binom{\mu+j+2r}{j} \cdot c_{2m+1,j} - \sum_{j=1}^{2r} c_{2m+1,j} - \sum_{j=2r+1}^{2m+1} 2 \cdot c_{2m+1,j} &= R^{\mu}_{2r+1,1}(2m+1). \end{aligned}$$

We encountered the following computational challenges:

- Creative telescoping for the summation in the second identity did not finish.
- In the third identity, a singularity appeared in the certificate $Q$ at $j = 1$ (for both summations) and we were not able to automatically certify our telescoper.
- The other "relationship" took even more computational resources due to the additional sum in the third identity.

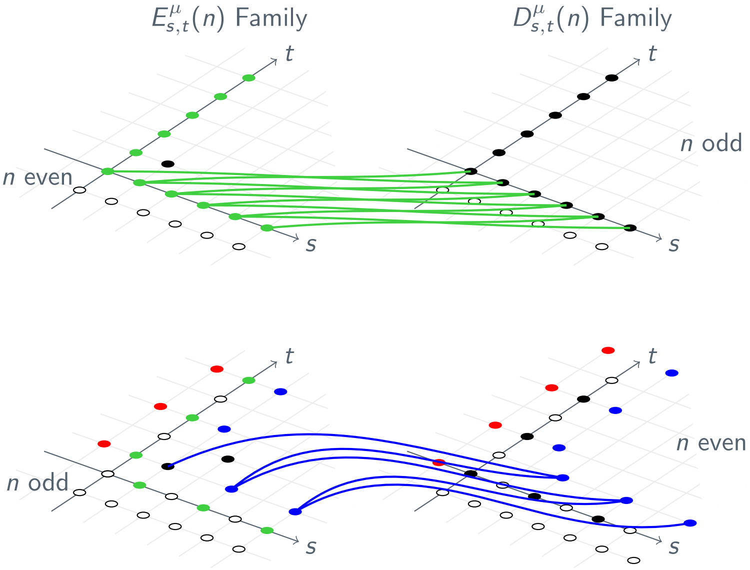
# The Big Picture



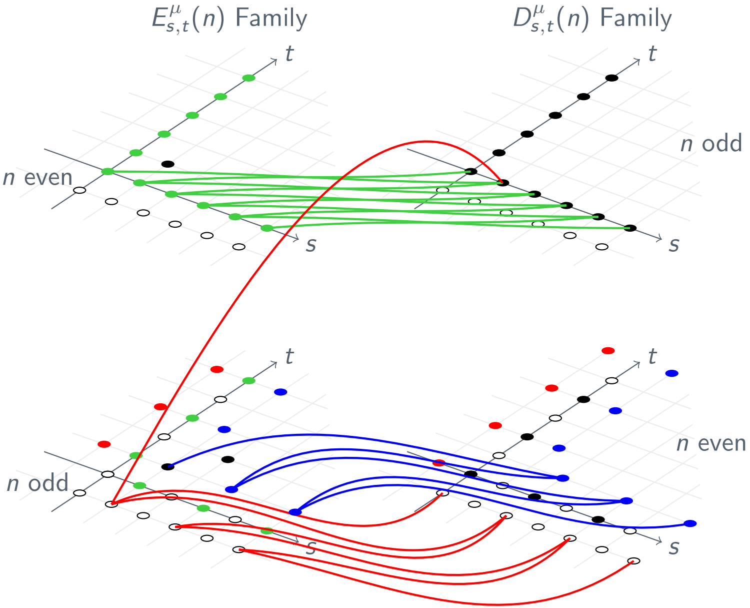


# The Big Picture



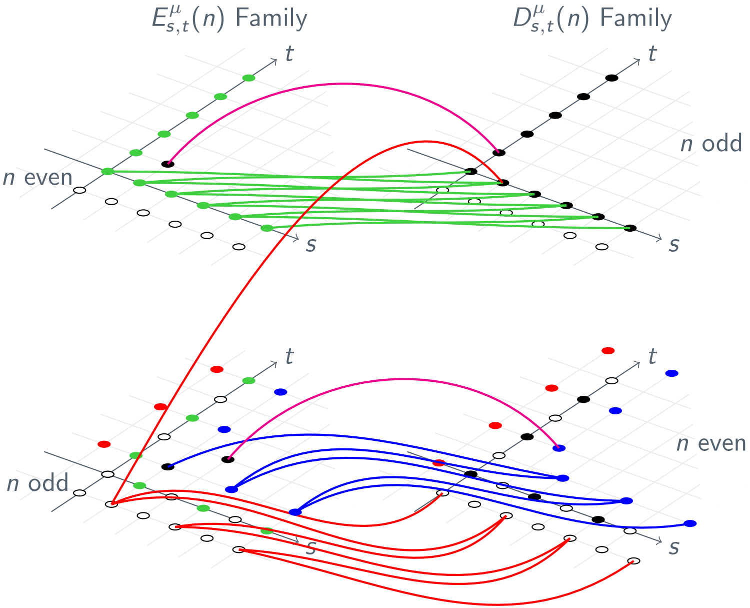


# The Big Picture



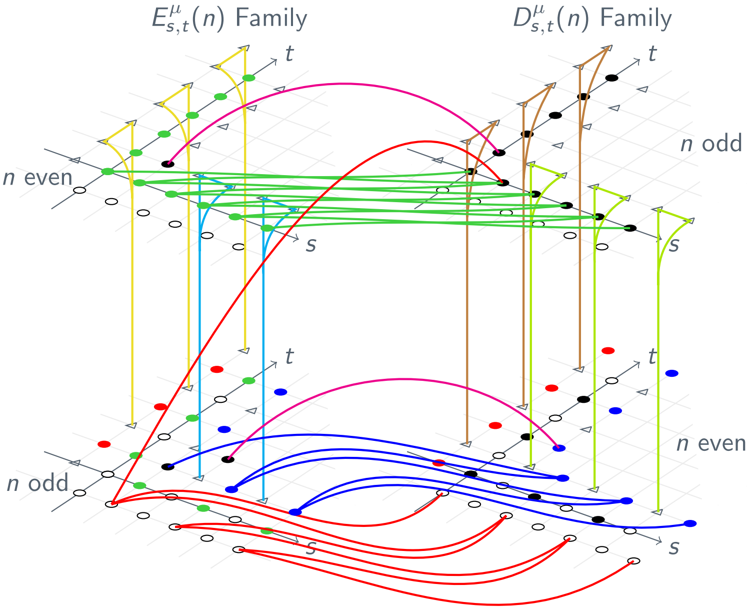
# The Big Picture





# The Big Picture





# Website



For paper and code, please visit:
HTTPS://WONGEY.GITHUB.IO/BINOM-DET/

Thank you!