The size of the minimal automaton for an algebraic sequence

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Algebraic sequences

$\mathbb{F}_q$ denotes the finite field with q elements.

Let $s(n)_{n\geq 0}$ be a sequence of elements in $\mathbb{F}_q$.

$s(n)_{n\geq 0}$ is algebraic if there exists a nonzero polynomial $P(x, y) \in \mathbb{F}_q[x, y]$ such that $P(x, \sum_{n\geq 0} s(n)x^n) = 0$.

Combinatorial motivation: Integer sequences modulo p.

Example

Catalan numbers $C(n)_{n\geq 0} = 1, 1, 2, 5, 14, 42, 132, 429, \ldots$

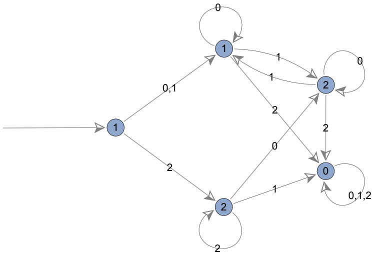
$F(x) = \sum_{n\geq 0} C(n)x^n$ satisfies $xy^2 - y + 1 = 0$ over $\mathbb{Q}$.

$F(x) = \sum_{n\geq 0}(C(n) \bmod 3)x^n$ satisfies $xy^2 + 2y + 1 = 0$ over $\mathbb{F}_3$.

Automatic sequences

A sequence $s(n)_{n \geq 0}$ is *q-automatic* there is an automaton that outputs $s(n)$ when fed the base-q digits of n.
Convention in this talk: start with the least significant digit.

This automaton computes $C(n)$ mod 3:



$C(9) = 4862 \equiv ?$ mod 3. Since $9 = 100_3$, $C(9) \equiv \boxed{2}$ mod 3.

$(C(n) \bmod 3)_{n \geq 0} = 1, 1, 2, 2, 2, 0, 0, 0, 2, 2, \ldots$ is 3-automatic.

Theorem (Christol 1979/1980)

A sequence $s(n)_{n\geq 0}$ of elements in $\mathbb{F}_q$ is algebraic if and only if it is q-automatic.

Two ways to represent such sequences: polynomials and automata.

How does the size of the automaton (number of states) depend on the x-degree (height) and y-degree (degree) of the polynomial?

Theorem (Bridy 2017)

Let $s(n)_{n\geq 0}$ be an algebraic sequence of elements in $\mathbb{F}_q$.
If its minimal polynomial has height h, degree d, and genus g, then the number of states in its minimal automaton is at most

$$(1 + o(1))q^{h+d+g-1},$$

where $o(1)$ tends to 0 as any of q, h, d, g gets large.

The genus satisfies $g \leq (h-1)(d-1)$; generically $g = (h-1)(d-1)$.

Corollary

The number of states is at most $(1 + o(1))q^{hd}$.

Can we get this bound without algebraic geometry? Yes.

Is the bound sharp? We suspect yes, but this is an open question.

How to construct an automaton?

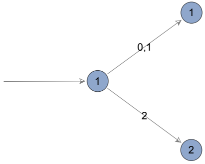
Let $r \in \{0, 1, \ldots, q-1\}$.
The Cartier operator Λ_r picks out every qth term, starting with $s(r)$:

$$\Lambda_r(s(n)_{n\geq 0}) := s(qn + r)_{n\geq 0}$$

Iteratively apply $\Lambda_0, \Lambda_1, \ldots, \Lambda_{q-1}$ to $s(n)_{n\geq 0}$.
Create one state in the automaton for each distinct sequence.

Let $s(n) = (C(n) \bmod 3)$. $\quad s(n)_{n\geq 0} = 1, 1, 2, 2, 2, 0, 0, 0, 2, \ldots$.

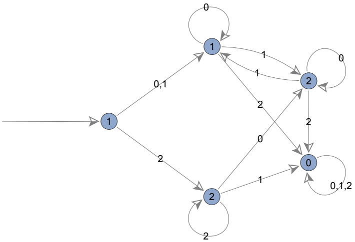
$\Lambda_0(s(n)_{n\geq 0}) = s(3n+0)_{n\geq 0} = 1, 2, 0, 2, 1, 0, 0, 0, 0, \ldots$ new!
$\Lambda_1(s(n)_{n\geq 0}) = s(3n+1)_{n\geq 0} = 1, 2, 0, 2, 1, 0, 0, 0, 0, \ldots = \Lambda_0(s(n)_{n\geq 0})$
$\Lambda_2(s(n)_{n\geq 0}) = s(3n+2)_{n\geq 0} = 2, 0, 2, 1, 0, 0, 0, 0, 2, \ldots$ new!



Label each state with the initial term of the corresponding sequence.

$\Lambda_0(\Lambda_0(s(n)_{n\geq 0})) = 1, 2, 0, 2, 1, 0, 0, 0, 0, 2, \ldots = \Lambda_0(s(n)_{n\geq 0})$
$\Lambda_1(\Lambda_0(s(n)_{n\geq 0})) = 2, 1, 0, 1, 2, 0, 0, 0, 0, 1, \ldots$ new!
$\Lambda_2(\Lambda_0(s(n)_{n\geq 0})) = 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots$ new!

$\Lambda_r(\Lambda_2(s(n)_{n\geq 0})) \quad \ldots$



A sequence is q-automatic if and only if this process terminates.

But we can't tell if sequences are equal from finitely many terms.

Use a different representation: diagonals of rational functions.

Theorem (Furstenberg 1967)

Let K be a field, and let $P(x,y) \in K[x,y]$ such that $\frac{\partial P}{\partial y}(0,0) \neq 0$. If $F(x) \in K[\![x]\!]$ satisfies $F(0) = 0$ and $P(x, F(x)) = 0$, then

$$F(x) = \mathcal{D}\left(\frac{y\frac{\partial P}{\partial y}(xy,y)}{P(xy,y)/y}\right).$$

The arguments xy arise from shearing the array of coefficients.

It will be more convenient to not shear. Then

$$F(x) = \mathcal{C}\left(\frac{y\frac{\partial P}{\partial y}(x,y)}{P(x,y)/y}\right)$$

where $\mathcal{C}$ projects a Laurent series to the column $\langle x^i y^0 : i \geq 0 \rangle$.

Example

$$\sum_{n\geq 0} (C(n) \bmod 3) x^n \text{ satisfies } xy^2 + 2y + 1 = 0 \text{ over } \mathbb{F}_3.$$

$$\sum_{n\geq 1} (C(n) \bmod 3) x^n \text{ is the } y^0 \text{ column of}$$

$$\begin{aligned}
\frac{y\frac{\partial P}{\partial y}(x,y)}{P(x,y)/y} = \frac{y(2xy+(2x+2))}{(xy^2+(2x+2)y+x)/y} = {} & 0x^0y^0 + 1x^0y^1 + 0x^0y^2 + 0x^0y^3 + 0x^0y^4 + 0x^0y^5 + \cdots \\
& + 0x^1y^{-1} + 1x^1y^0 + 0x^1y^1 + 2x^1y^2 + 0x^1y^3 + 0x^1y^4 + \cdots \\
& + 0x^2y^{-2} + 1x^2y^{-1} + 2x^2y^0 + 0x^2y^1 + 1x^2y^2 + 2x^2y^3 + \cdots \\
& + 0x^3y^{-3} + 1x^3y^{-2} + 1x^3y^{-1} + 2x^3y^0 + 0x^3y^1 + 1x^3y^2 + \cdots \\
& + 0x^4y^{-4} + 1x^4y^{-3} + 0x^4y^{-2} + 2x^4y^{-1} + 2x^4y^0 + 0x^4y^1 + \cdots \\
& + 0x^5y^{-5} + 1x^5y^{-4} + 2x^5y^{-3} + 0x^5y^{-2} + 0x^5y^{-1} + 0x^5y^0 + \cdots \\
& + \cdots .
\end{aligned}$$

We have embedded $s(n)_{n\geq 0}$ into a bivariate series $\frac{S_0}{Q}$ where $Q = P/y$.
Can we compute $\Lambda_r(s(n)_{n\geq 0})$?

Define

$$\Lambda_r(x^n) = \begin{cases} x^{\frac{n-r}{q}} & \text{if } n \equiv r \mod q \\ 0 & \text{otherwise} \end{cases}$$

and extend linearly to power series. Define $\Lambda_{r,s}$ analogously.

The map

$$\lambda_{r,0}(S) := \Lambda_{r,0}\left(SQ^{q-1}\right)$$

on $\mathbb{F}_q[x, y]$ contains all information about $s \mapsto \Lambda_r(s)$ (and some extra):

$$\Lambda_r \mathcal{C}\left(\frac{S}{Q}\right) = \mathcal{C}\left(\frac{\Lambda_{r,0}\left(SQ^{q-1}\right)}{Q}\right)$$

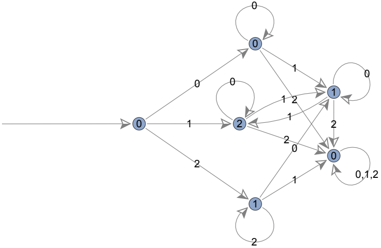
We construct an automaton by iterating $\lambda_{0,0}, \ldots, \lambda_{q-1,0}$.

$(C(n) \bmod 3)_{n \geq 1}$ is a column of $\frac{S_0}{Q} := \frac{y(2xy+2x+2)}{(xy^2+(2x+2)y+x)/y}$.

$\lambda_{0,0}(S_0) = xy + x$ new!
$\lambda_{1,0}(S_0) = 2$ new!
$\lambda_{2,0}(S_0) = y + 1$ new!

$\lambda_{0,0}(xy + x) = xy + x = \lambda_{0,0}(S_0)$ ...

If two polynomials are equal, the corresponding sequences are equal.



The automaton may not be minimal.

Let $V := \langle x^i y^j : 0 \le i \le h \text{ and } 0 \le j \le d-1 \rangle$. $\dim V = (h+1)d$

Proposition

For each $r \in \{0, 1, \dots, q-1\}$, we have $\lambda_{r,0}(S_0) \in V$.
For each $r \in \{1, \dots, q-1\}$,

$$\lambda_{r,0}(V) \subseteq \left\langle x^i y^j : 0 \le i \le h-1 \text{ and } 0 \le j \le d-1 \right\rangle$$

which has dimension hd.

Corollary:
The constructed automaton has at most $q^{hd} + |\mathrm{orb}_{\lambda_{0,0}}(S_0)|$ states.

It remains to bound $|\mathrm{orb}_{\lambda_{0,0}}(S_0)|$.

Certain orders of the basis for V show that $\lambda_{0,0}$ is highly structured.

Example

Let $q = 3$, $h = 2$, $d = 4$, and

$$P = (x^2 + x + 2)y^4 + xy^3 + (2x + 1)y^2 + (x^2 + 1)y + 2x^2 + x.$$

Basis:

$$\left(x^1y^0, x^1y^1, x^1y^2, \quad x^0y^1, x^0y^2, \quad x^0y^0, \quad x^1y^3, \quad x^0y^3, \quad x^2y^0, x^2y^1, x^2y^2, \quad x^2y^3\right).$$

Matrix for $\lambda_{0,0}$:

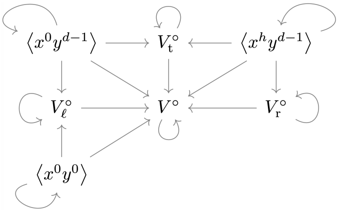
$$\begin{bmatrix}
1 & 1 & 1 & 2 & 1 & 2 & 0 & 0 & 2 & 2 & 0 & 0 \\
1 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 1 & 2 \\
2 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 1 & 1 & 1 \\
 & & & 1 & 2 & 1 & & 1 & & & & \\
 & & & 0 & 1 & 1 & & 1 & & & & \\
 & & & & & 1 & & & & & & \\
 & & & & & & 2 & 2 & & & & 1 \\
 & & & & & & & 1 & & & & \\
 & & & & & & & & 1 & 1 & 1 & 0 \\
 & & & & & & & & 2 & 1 & 0 & 1 \\
 & & & & & & & & 1 & 0 & 0 & 2 \\
 & & & & & & & & & & & 1
\end{bmatrix}$$

Basis of V:

x^0y^{d-1}	x^1y^{d-1}	$\cdots$	$x^{h-1}y^{d-1}$	x^hy^{d-1}
x^0y^{d-2}	x^1y^{d-2}	$\cdots$	$x^{h-1}y^{d-2}$	x^hy^{d-2}
$\vdots$	$\vdots$	$\iddots$	$\vdots$	$\vdots$
x^0y^1	x^1y^1	$\cdots$	$x^{h-1}y^1$	x^hy^1
x^0y^0	x^1y^0	$\cdots$	$x^{h-1}y^0$	x^hy^0

Theorem

Under applications of $\lambda_{0,0}$ on V, information flows as follows.



The left, right, and top subspaces are affected only by themselves.
Since $|V^\circ| = q^{(h-1)(d-1)}$, we show the borders contribute $\leq q^{h+d-1}$.

The left, right, and top subspaces are essentially univariate.

Fix $R \in \mathbb{F}_q[z]$. How big are orbits under $\lambda_0(S) := \Lambda_0\big(SR^{q-1}\big)$?
This is "just" a linear transformation.

Example

Let $q = 3$ and $R = (z^2 + 1)(z^3 + z^2 + 2) \in \mathbb{F}_3[z]$.
Compute $\mathrm{orb}_{\lambda_0}(S)$ from each $S \in \mathbb{F}_3[z]$ with $\deg S \leq \deg R$.
Period lengths that occur: $\{1, 2, 3, 6\}$

Example

Let $q = 3$ and $R = (z^2 + 1)(z^4 + z + 2) \in \mathbb{F}_3[z]$.
Period lengths: $\{1, 2, 4\}$

Consider all polynomials R with fixed degree.
Surprising fact: The maximal period length doesn't depend on q.

Theorem

Let $R \in \mathbb{F}_q[z]$ such that $R \neq 0$, $z \nmid R$, and R is square-free. Let $cR_1 \cdots R_m$ be its factorization into irreducibles, and let

$$\ell = \operatorname{lcm}(\deg R_1, \dots, \deg R_m).$$

Then $\lambda_0^\ell(S) = S$ for all $S \in \mathbb{F}_q[z]$ with $\deg S \leq \deg R$.

The upper bound is achieved when ℓ is maximized, subject to $\deg R_1 + \cdots + \deg R_m = \deg R$.

The Landau function $L(n)$ is the maximum value of $\operatorname{lcm}(n_1, \dots, n_m)$ over all integer partitions $(n_1, \dots, n_m)$ of n. Also arises in Bridy's proof.

Corollary

The number of states is at most

$$q^{hd} + q^{(h-1)(d-1)} L(h) L(d)^2 + \left\lceil \log_q \max(h, d, q) \right\rceil.$$

Asymptotically…

Landau (1903): $\log L(n) \sim \sqrt{n \log n}$

Massias–Nicolas–Robin (1988): $L(n) \le e^{(1+o(1))\sqrt{n \log n}}$

Corollary

The number of states is at most $(1 + o(1))q^{hd}$.

Example

The factor $1 + o(1)$ cannot be removed. Let $q = 2$ and

$$P = (x^3 + x^2 + 1)y^3 + (x^3 + 1)y^2 + (x^3 + x^2 + x + 1)y + x^3 + x^2$$

with $h = 3$ and $d = 3$. The number of states is $532 > 512 = q^{hd}$.

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