Combinatorial Exploration

a new approach to enumeration

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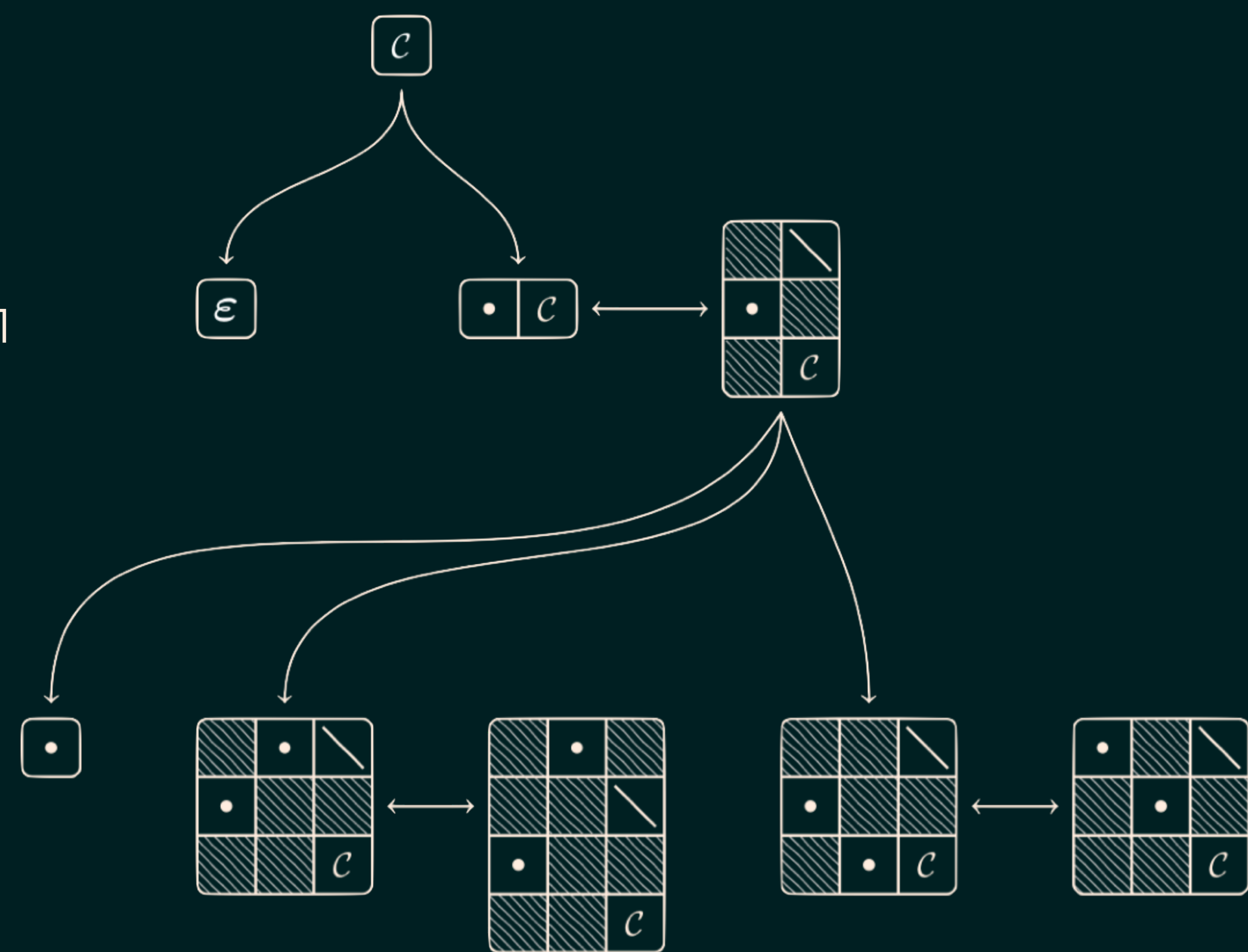
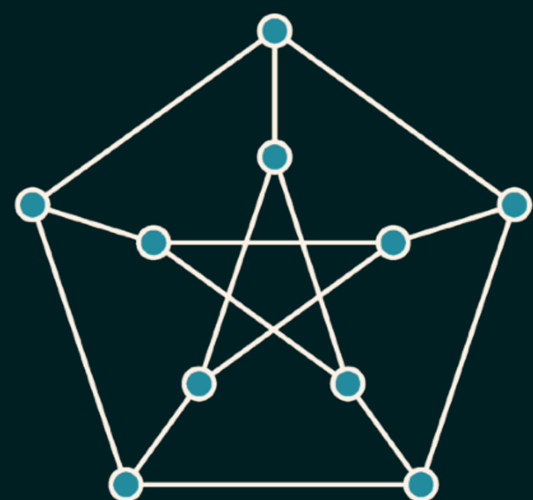
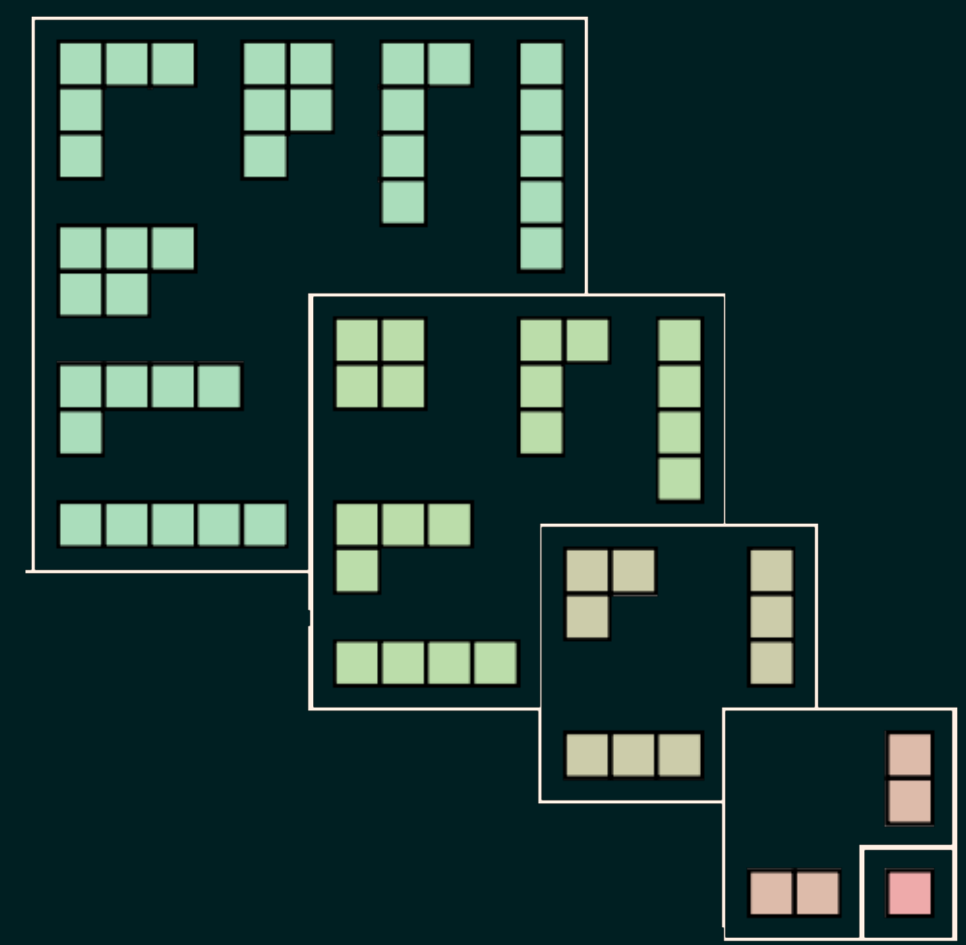


- Structure in combinatorial objects

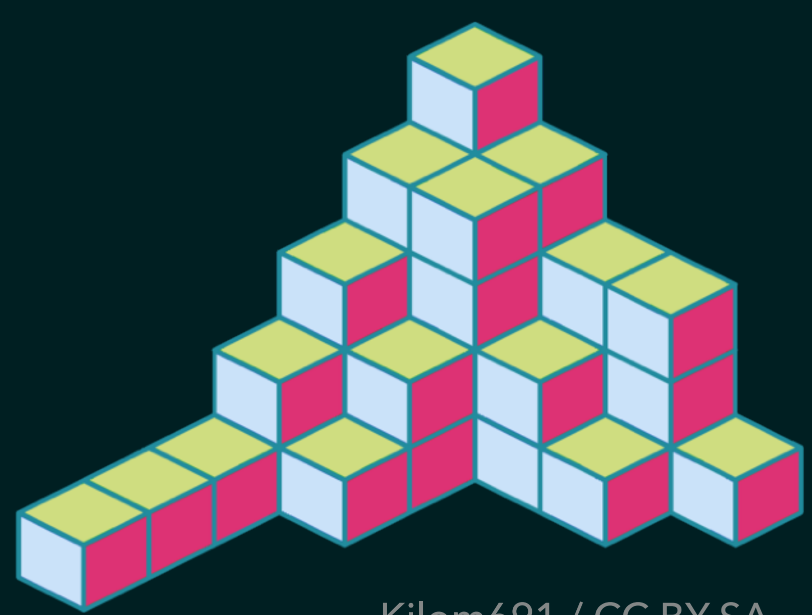
Graphs:



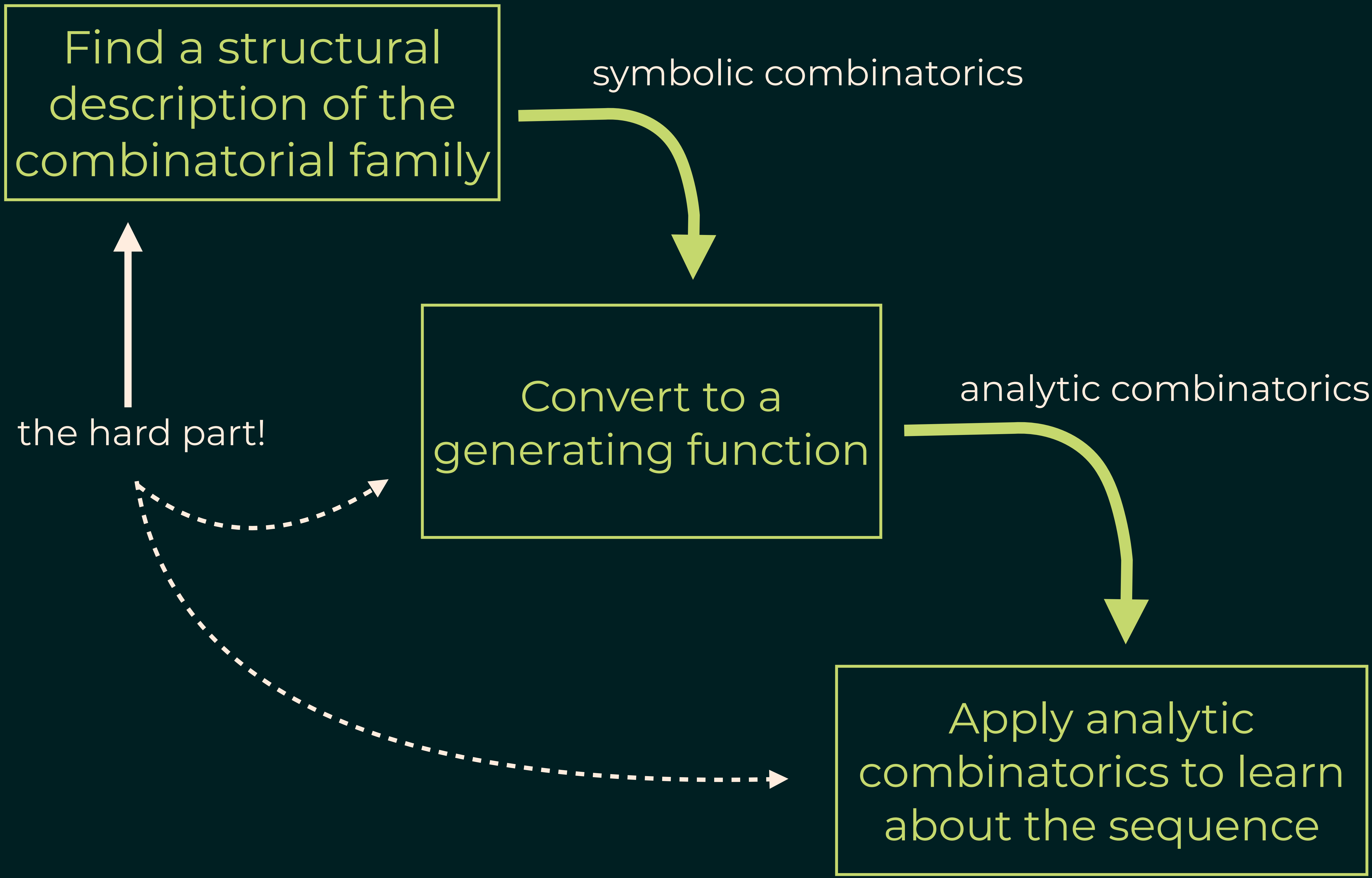
Integer
Partitions:



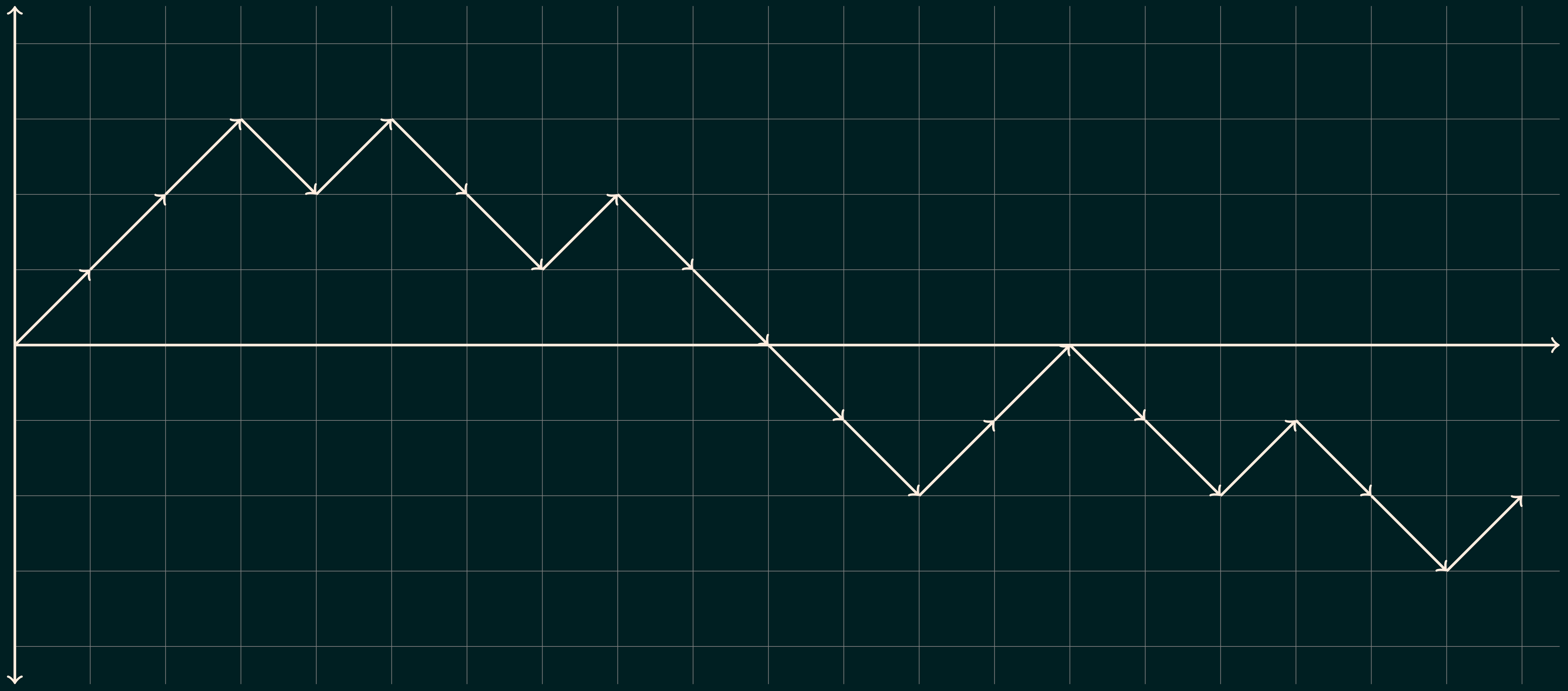
Plane
Partitions:



- Questions:
  - How many are there of each size?
    - explicit formula, generating function, polynomial-time algorithm
  - How does the counting sequence grow asymptotically as $n \to \infty$?
  - How can I sample an object of size n uniformly at random?
  - How can I build the objects of size n from the objects of smaller size?


Find a structural description of the combinatorial family
symbolic combinatorics
Convert to a generating function
analytic combinatorics
Apply analytic combinatorics to learn about the sequence
the hard part!

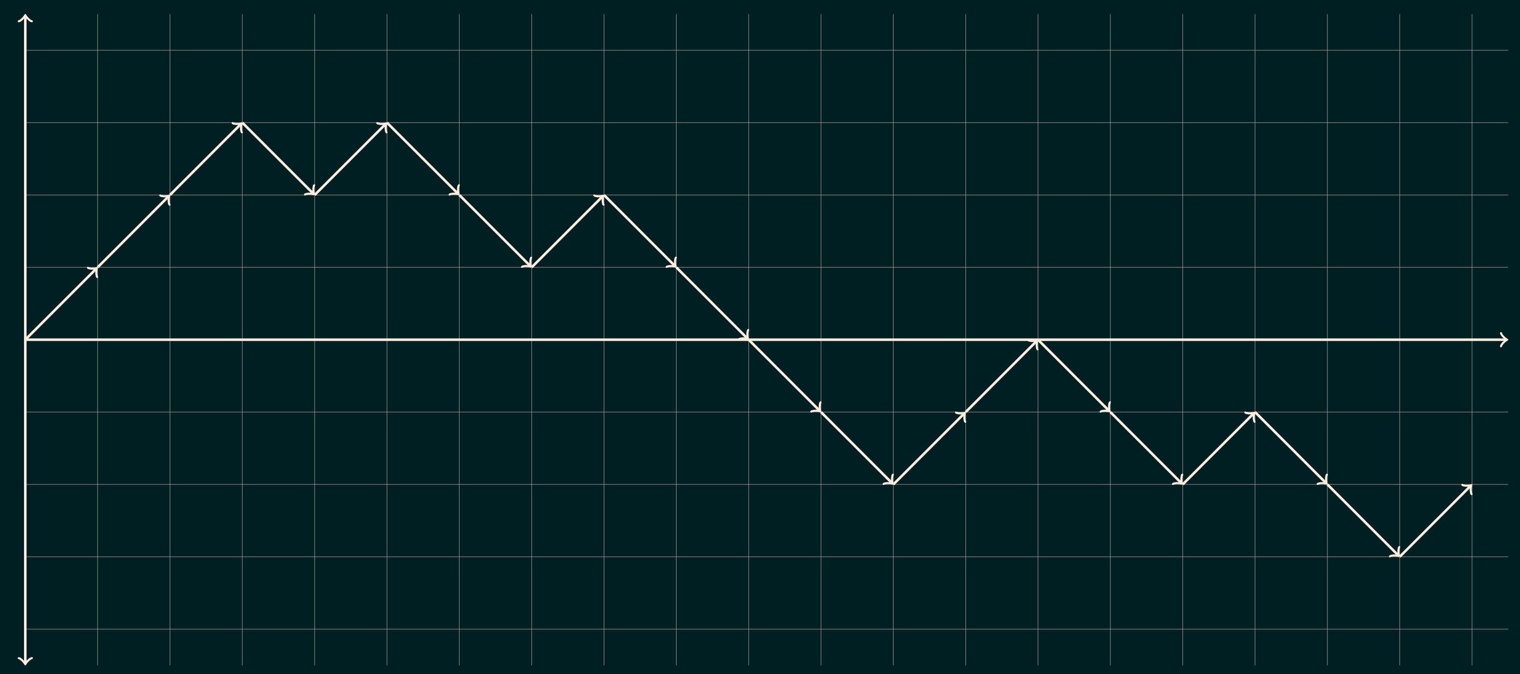
An *up-down walk* is a walk in the plane that starts at the origin and takes only NE and SE steps.



size = # of steps = 20

Before we ask questions, we need to understand the structure.

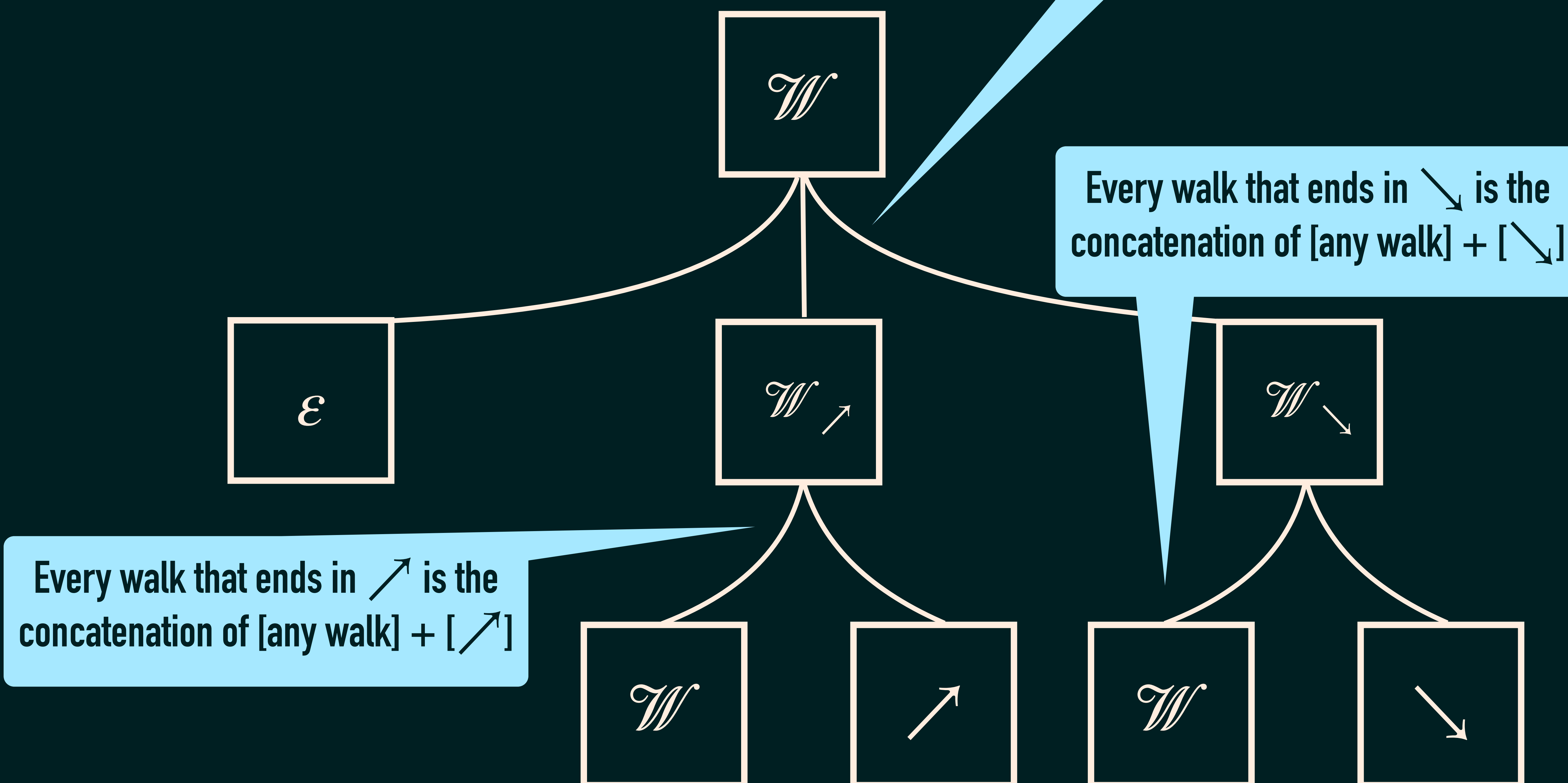
- The set of up-down walks of size n can be built by appending either a NE step or a SE step to every up-down walk of size $n-1$.
- Let's write this structural description in a tree format.



Structural description:

Let $\mathscr{W}$ be the set of up-down walks.

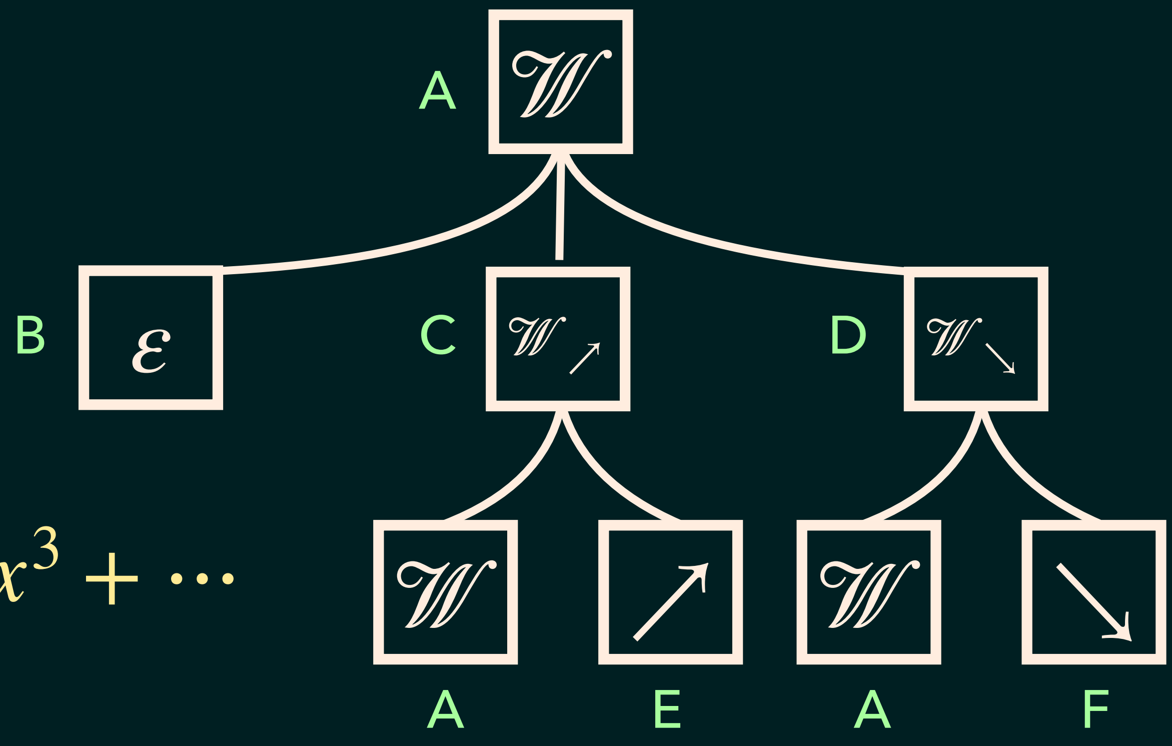
Every walk is either empty, or ends with ↗ or ends with ↘.



What do we learn from this structural decomposition?

Systems of equations for generating functions!

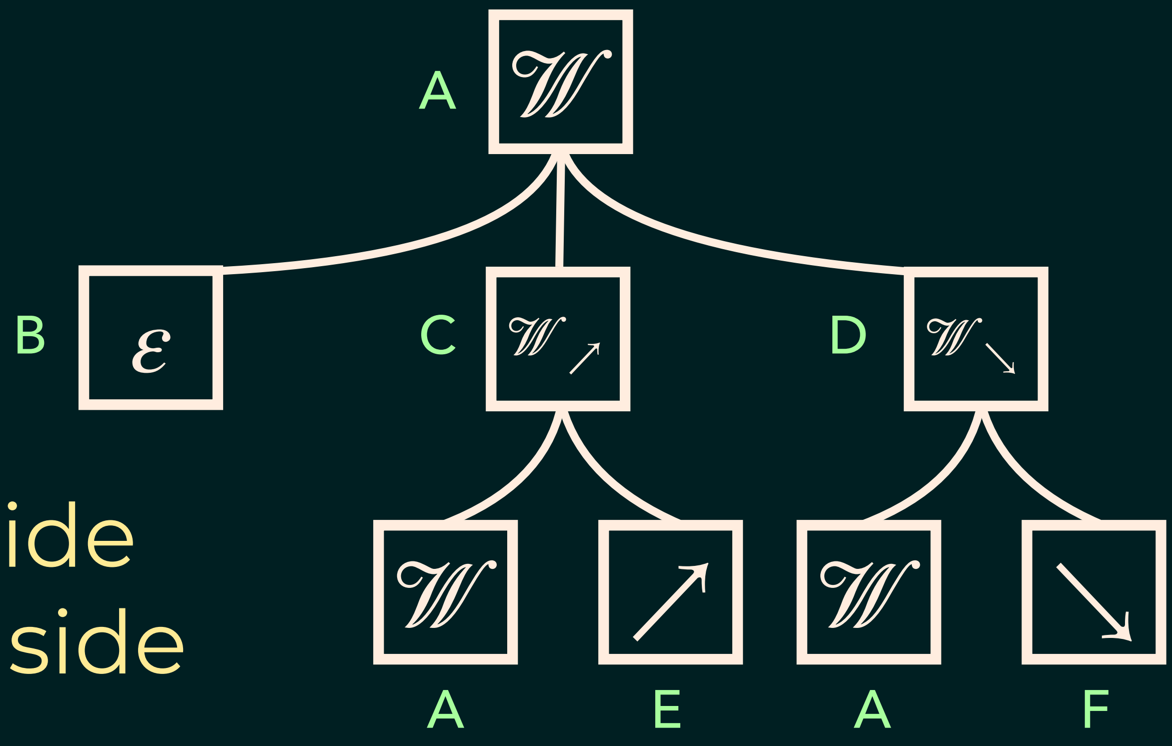
$$A(x) = B(x) + C(x) + D(x)$$
$$B(x) = 1$$
$$C(x) = A(x)E(x)$$
$$D(x) = A(x)F(x)$$
$$E(x) = x$$
$$F(x) = x$$



$$\implies A(x) = \frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \cdots$$

These structural description trees are just a pictorial way to represent a combinatorial specification.

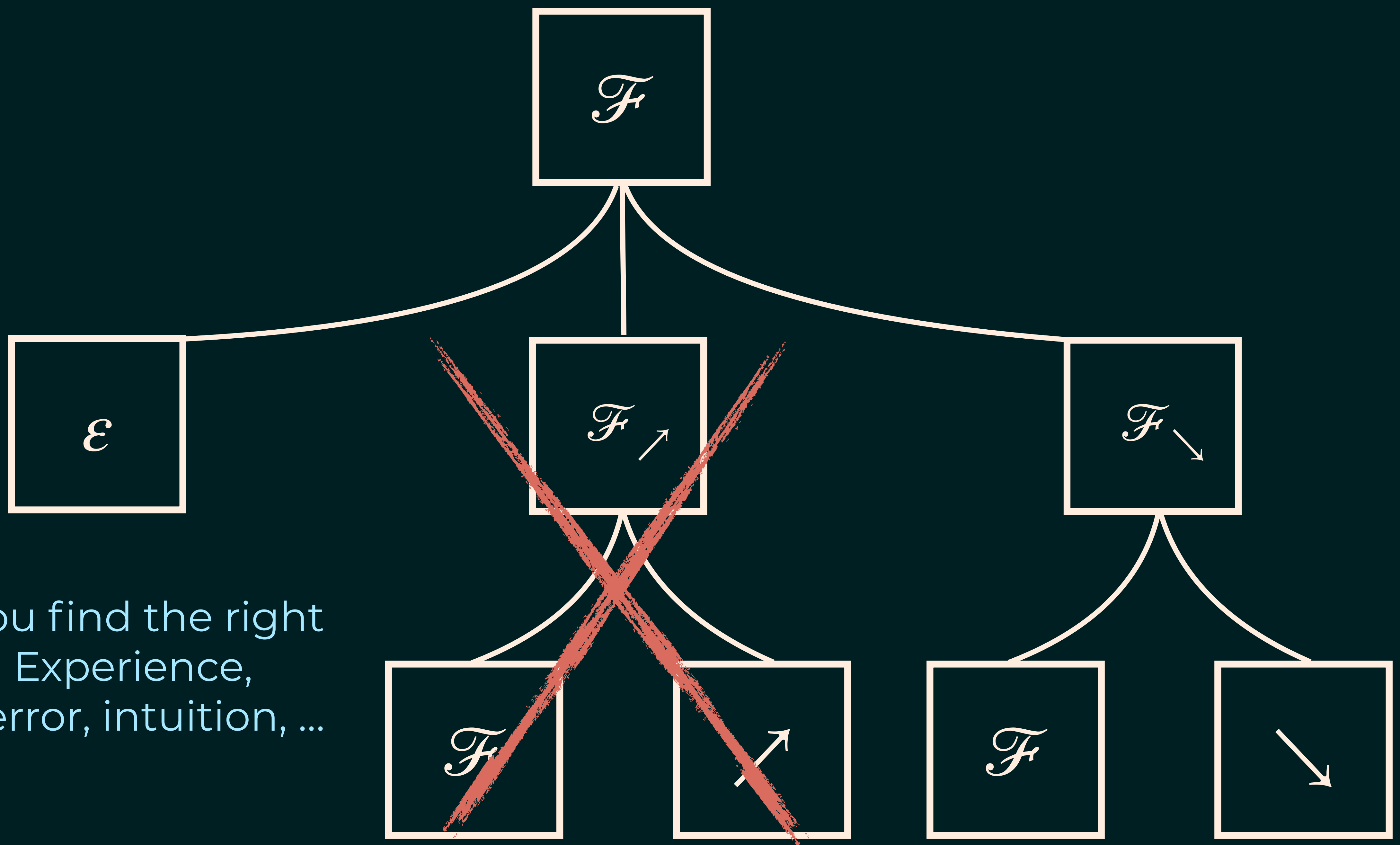
$$
\begin{aligned}
A &\to (B, C, D) \\
B &\to \{\varepsilon\} \\
C &\to (A, E) \\
D &\to (A, F) \\
E &\to \{ \nearrow \} \\
F &\to \{ \searrow \}
\end{aligned}
$$



every symbol on the right-hand side appears on exactly one left-hand side

Slightly more complicated:

$\mathscr{F}$ = the set of walks that don't go up three times in a row



How do you find the right structure? Experience, trial-and-error, intuition, …

Find a structural description of the set of objects

symbolic combinatorics

let's automate this part

Convert to a generating function

analytic combinatorics

Apply analytic combinatorics to learn about the sequence

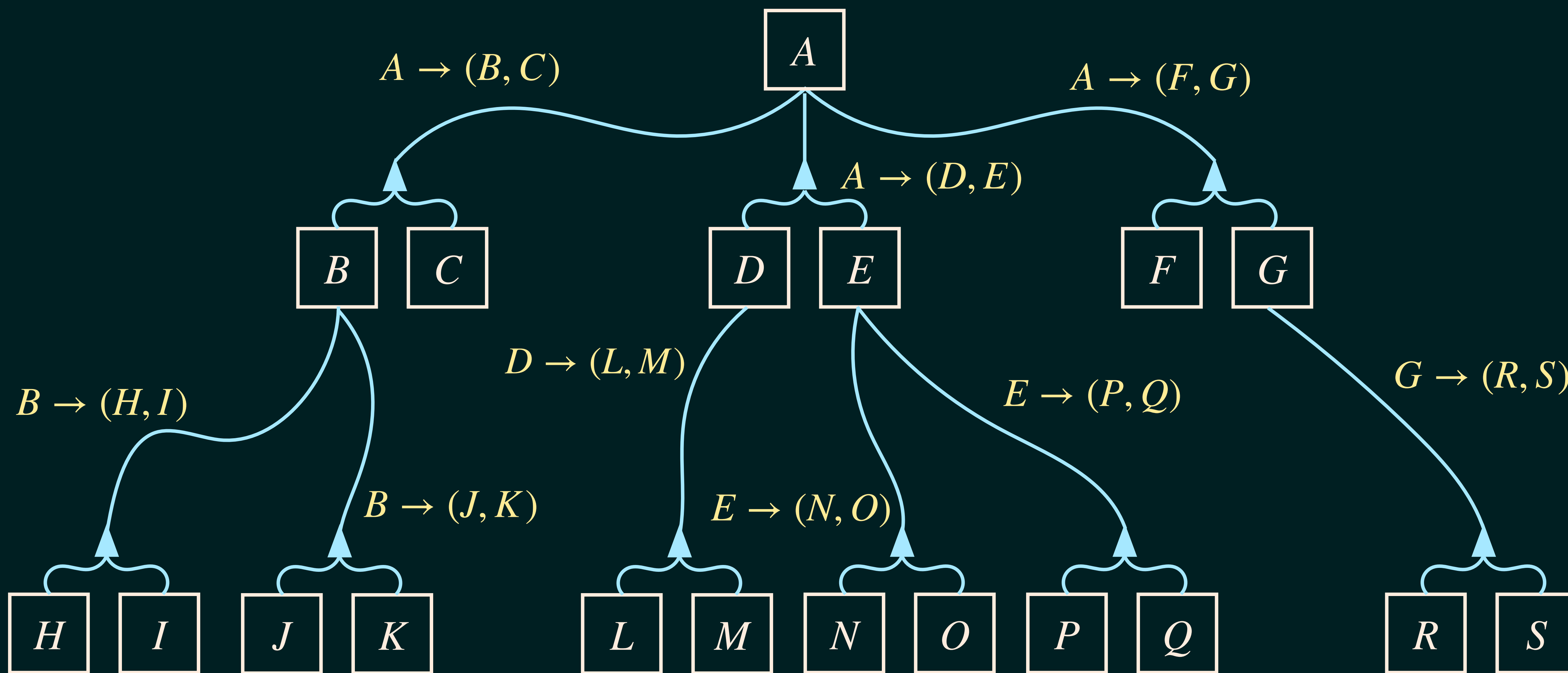
Requirements:

- a domain of all objects (up-down walks)
- a representation for the sets of objects that you'll be working with ("$\mathcal{W}_{\nearrow}$" is the set of up-down walks that end with $\nearrow$)
- decomposition strategies to split the sets into (hopefully) simpler sets

Procedure:

- start with a subset of the domain that you want to understand (up-down walks that don't go up three times in a row)
- try to apply all decomposition strategies to it
- then apply all decomposition strategies to the new (hopefully) simpler sets, and repeat
- stop when you understand all the parts

develop strategies for a whole domain → apply them to subsets of the domain you want to learn about



this is just a pictorial version of a list of combinatorial rules

when the giant list of rules you're generating contains a subset that is a combinatorial specification, you win!

To run Combinatorial Exploration on a new type of object, you just need to:

- decide on a good way to represent sets of those objects, and write a Python class for it
- decide on effective decomposition strategies (this is where domain-specific experience comes in handy)
- plug these right into our framework, and hit go

- ~ 50k lines of Python code

https://github.com/PermutaTriangle/comb_spec_searcher

Domains we've coded:

- permutation patterns (inspired this work)
- set partitions
- Motzkin paths

Domains that seem promising on paper:

- polyominoes
- inversion sequences
- alternating sign matrices

Given a set of permutations B, you can study the set of permutations avoiding the permutations in B as patterns — these sets are called permutation classes.

For the cases where B contains two permutations of length 4, there are essentially 56 different permutation classes.

(https://en.wikipedia.org/wiki/Enumerations_of_specific_permutation_classes)

Their enumerations are all known now, but it took several decades and dozens of papers.

Combinatorial Exploration can enumerate all of them.

Computational Difficulties

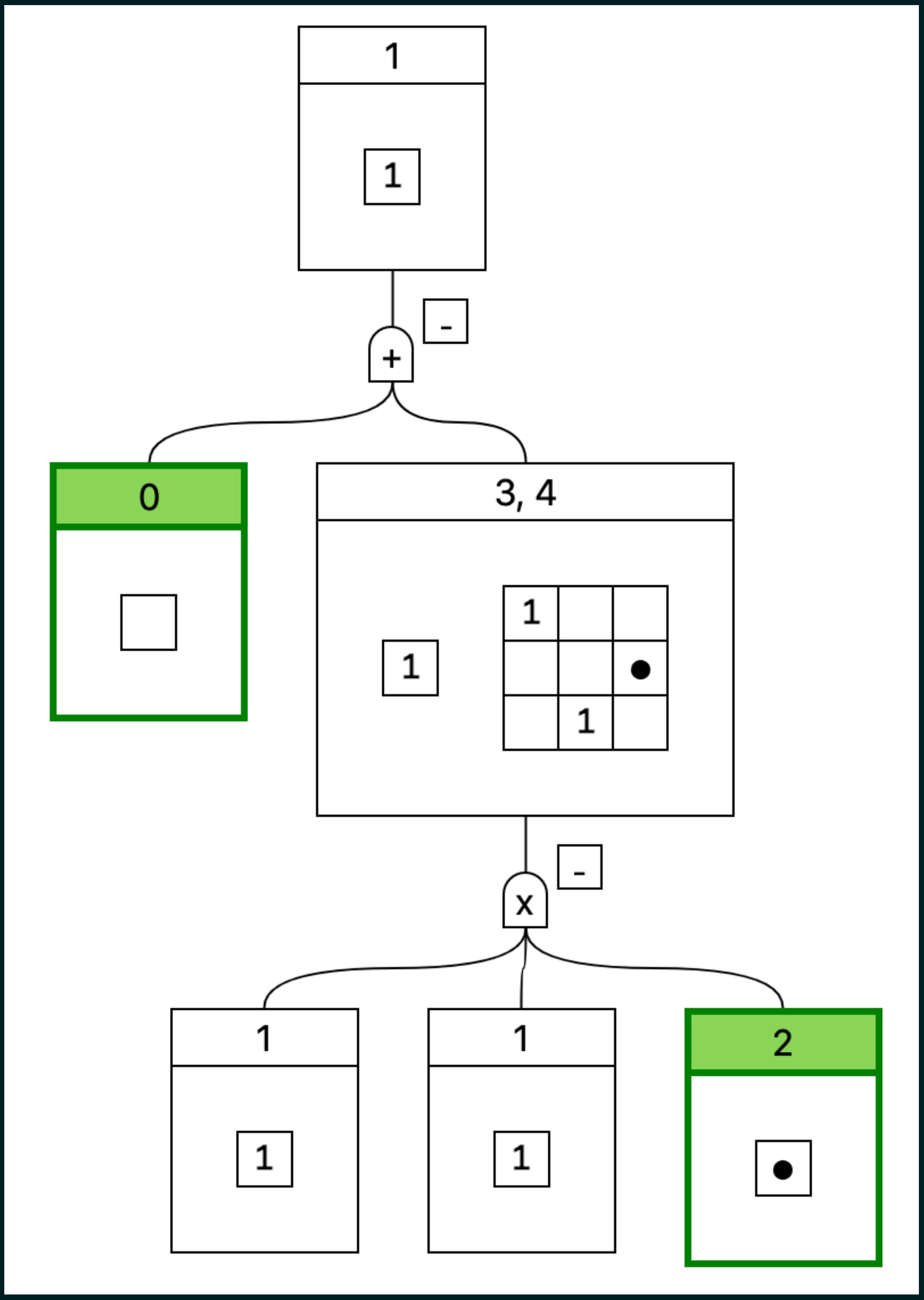
Permutations avoiding 132:

$$F_0(x) = F_1(x) + F_2(x)$$
$$F_1(x) = F_0(x)^2 \cdot F_3(x)$$
$$F_2(x) = 1$$
$$F_3(x) = x$$

Permutations avoiding 1432 and 2143:

$$F_0(x) = F_{547}(x) + F_{373}(x)$$
$$F_1(x) = F_0(x) - F_{118}(x)$$
$$\cdots$$
$$F_{549}(x) = 0$$

550 equations $\longrightarrow$ guess-and-check


1
1
-
+
0
3, 4
1
1
1
-
x
1
1
1
1
2

<u>Computational Difficulties</u> — with 1 catalytic variable!

Permutations avoiding 123:

$$
\begin{aligned}
F_0(x) &= F_{11}(x) + F_6(x) \\
F_1(x) &= F_{12}(x) \cdot F_2(x) \\
&\boxed{F_2(x) = F_3(x,1)} \\
F_3(x,y) &= F_7(x,y) + F_8(x,y) \\
F_4(x,y) &= F_{12}(x) \cdot F_5(x,y) \cdot F_8(x,y) \\
&\boxed{F_5(x,y) = \frac{yF_3(x,y) - F_3(x,1)}{y-1}} \\
F_6(x) &= F_1(x) \\
F_7(x,y) &= F_4(x,y) \\
F_8(x,y) &= F_{10}(x,y) + F_{11}(x) \\
F_9(x,y) &= F_{13}(x,y) \cdot F_8(x,y) \\
F_{10}(x,y) &= F_9(x,y) \\
F_{11}(x) &= 1 \\
F_{12}(x) &= x \\
F_{13}(x,y) &= xy
\end{aligned}
$$

Computational Difficulties — with 1 catalytic variable!

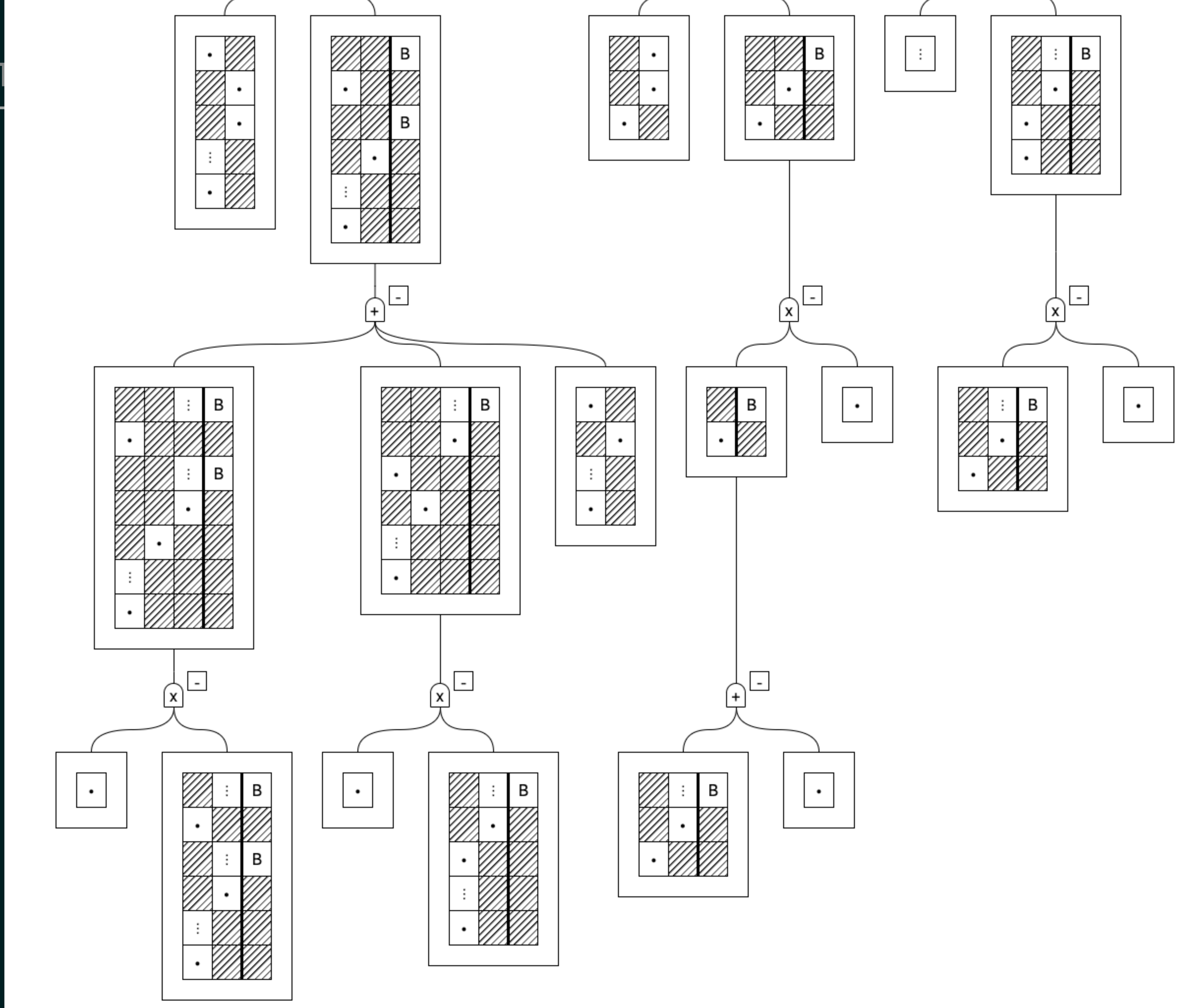
Much harder with hundreds of equations (intermediate computations become huge before simplifying again).

Is there a guess-and-check approach that could work?

Computational Difficulties — with 2+ catalytic variables!

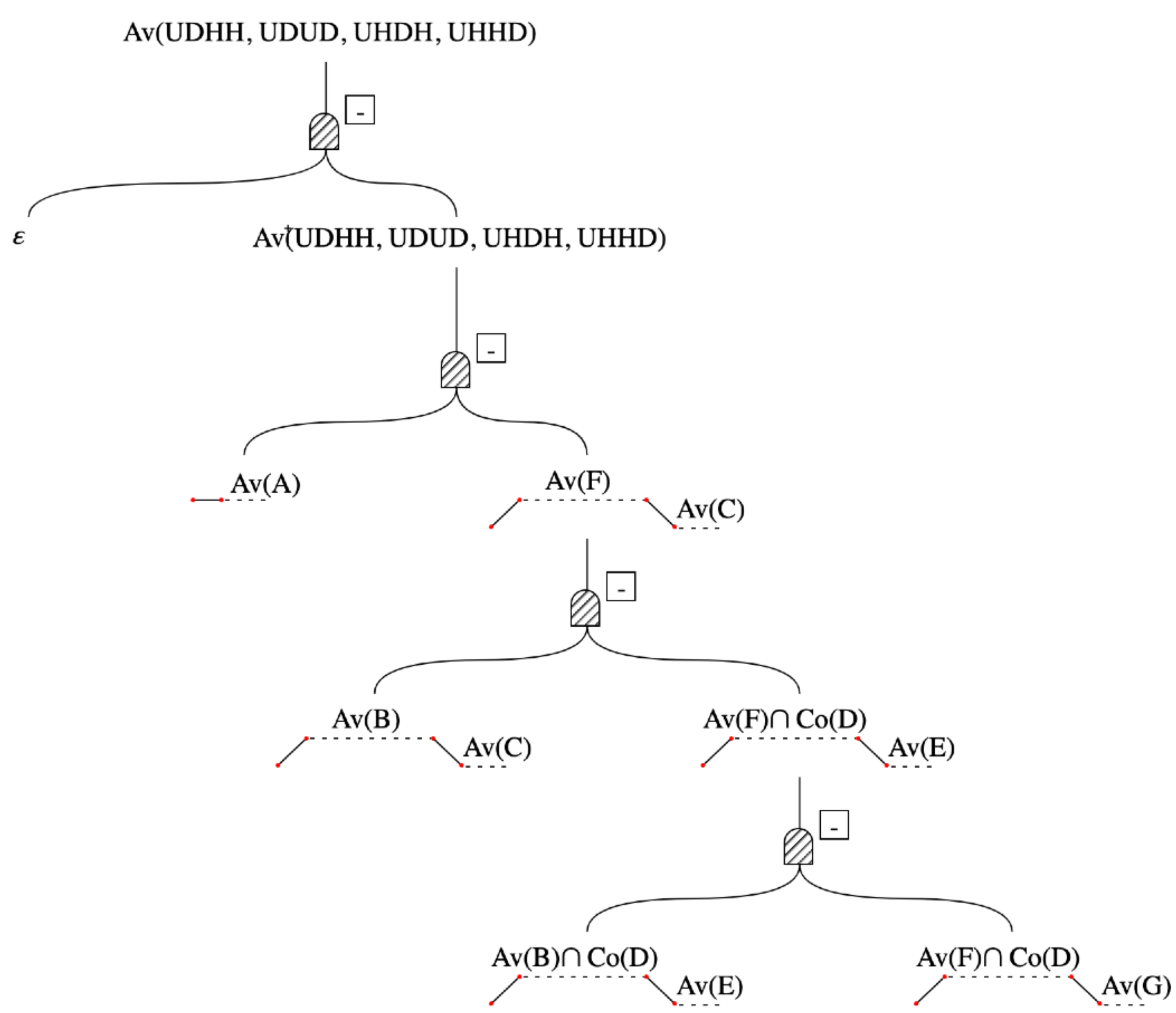
$$F_0(x) = F_1(x) + F_{15}(x)$$
$$F_1(x) = F_{16}(x) \cdot F_2(x)$$
$$F_2(x) = F_3(x,1)$$
$$F_3(x,y) = F_{12}(x,y) + F_{15}(x) + F_4(x,y)$$
$$F_4(x,y) = F_{17}(x,y) \cdot F_5(x,y)$$
$$\boxed{F_5(x,y) = F_{14}(x,1,y)}$$
$$F_6(x,y,z) = F_{11}(x,y,z) + F_{15}(x) + F_7(x,y,z) + F_9(x,y,z)$$
$$F_7(x,y,z) = F_{17}(x,z) \cdot F_8(x,y,z)$$
$$\boxed{F_8(x,y,z) = \frac{yF_{14}\left(x, \frac{y}{z}, z\right) - z \cdot F_{14}(x,1,z)}{y-z}}$$
$$F_9(x,y,z) = F_{10}(x,y,z) \cdot F_{16}(x)$$

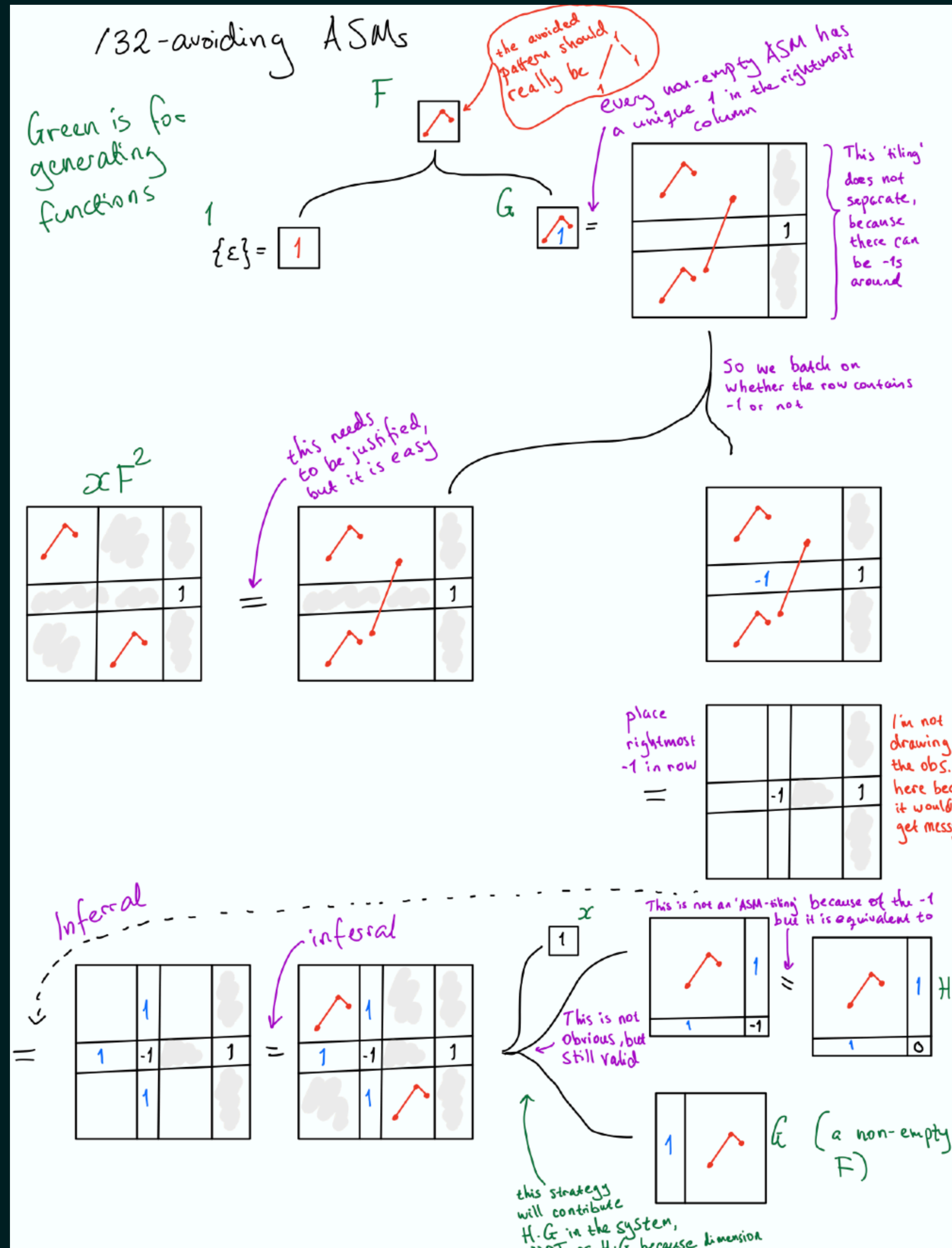
$$F_{10}(x,y,z) = \frac{zF_6(x,y,z) - F_6(x,y,1)}{z-1}$$
$$F_{11}(x,y,z) = F_{17}(x,y) \cdot F_6(x,y,z)$$
$$F_{12}(x,y) = F_{13}(x,y) \cdot F_{16}(x)$$
$$F_{13}(x,y) = \frac{yF_3(x,y) - F_3(x,1)}{y-1}$$
$$\boxed{F_{14}(x,y,z) = F_6(x,yz,z)}$$
$$F_{15}(x) = 1$$
$$F_{16}(x) = x$$
$$F_{17}(x,y) = xy$$



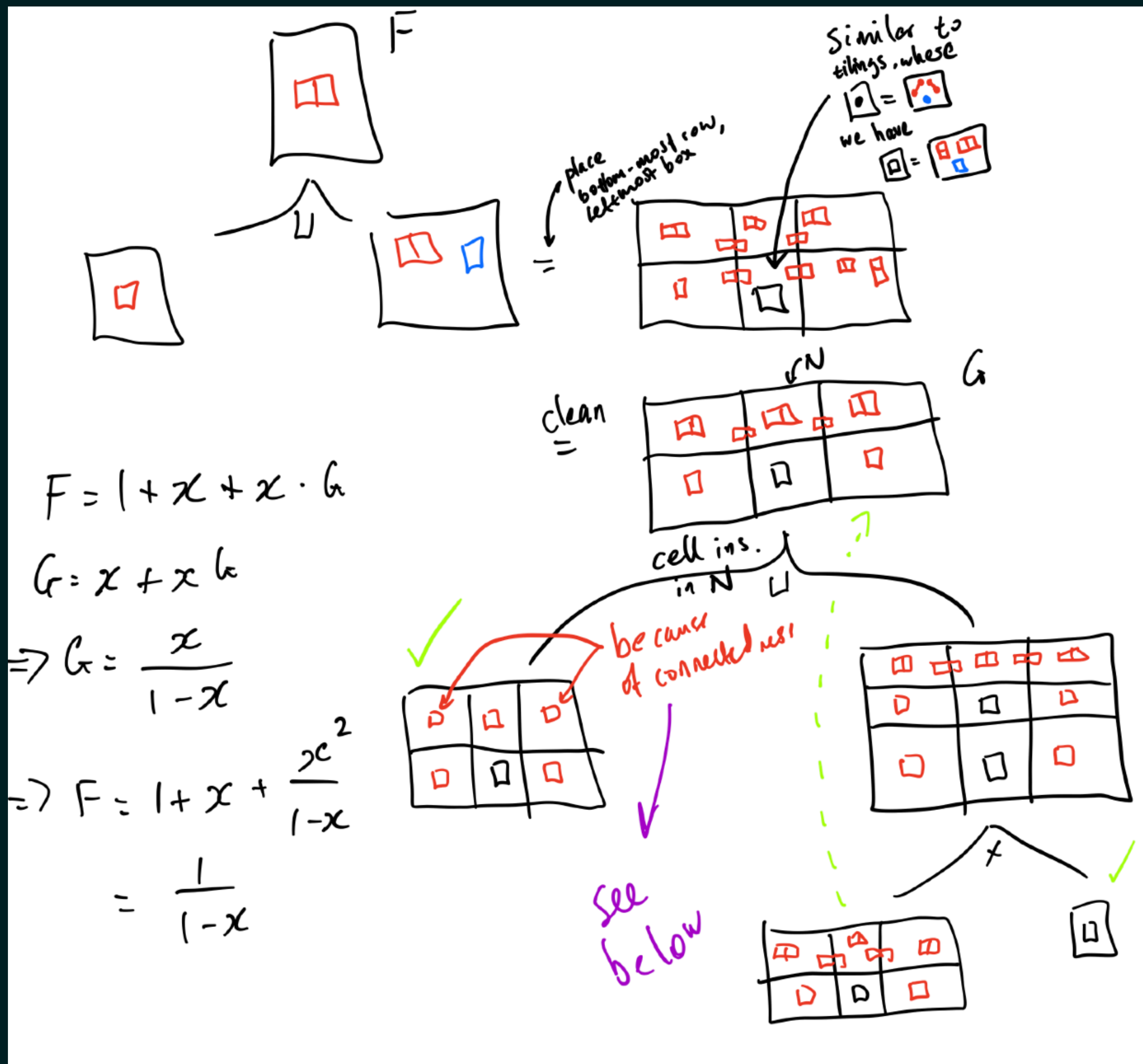
Motzkins paths avoiding: UDHH, UDUD, UHDH, UHHD

Proof tree for Motzkin paths avoiding: UDHH, UDUD, UHDH, UHHD





132-avoiding ASMs
Green is for generating functions
F
the avoided pattern should really be
every non-empty ASM has a unique 1 in the rightmost column
1
{ε} =
G
This 'tiling' does not separate, because there can be -1s around
So we batch on whether the row contains -1 or not
xF²
this needs to be justified, but it is easy
place rightmost -1 in row
I'm not drawing the obs. here bec. it would get messy
Inferral
inferral
x
This is not an 'ASM-tiling' because of the -1 but it is equivalent to
H
This is not obvious, but still valid
G (a non-empty F)
this strategy will contribute H·G in the system, NOT x·H·G because dimension



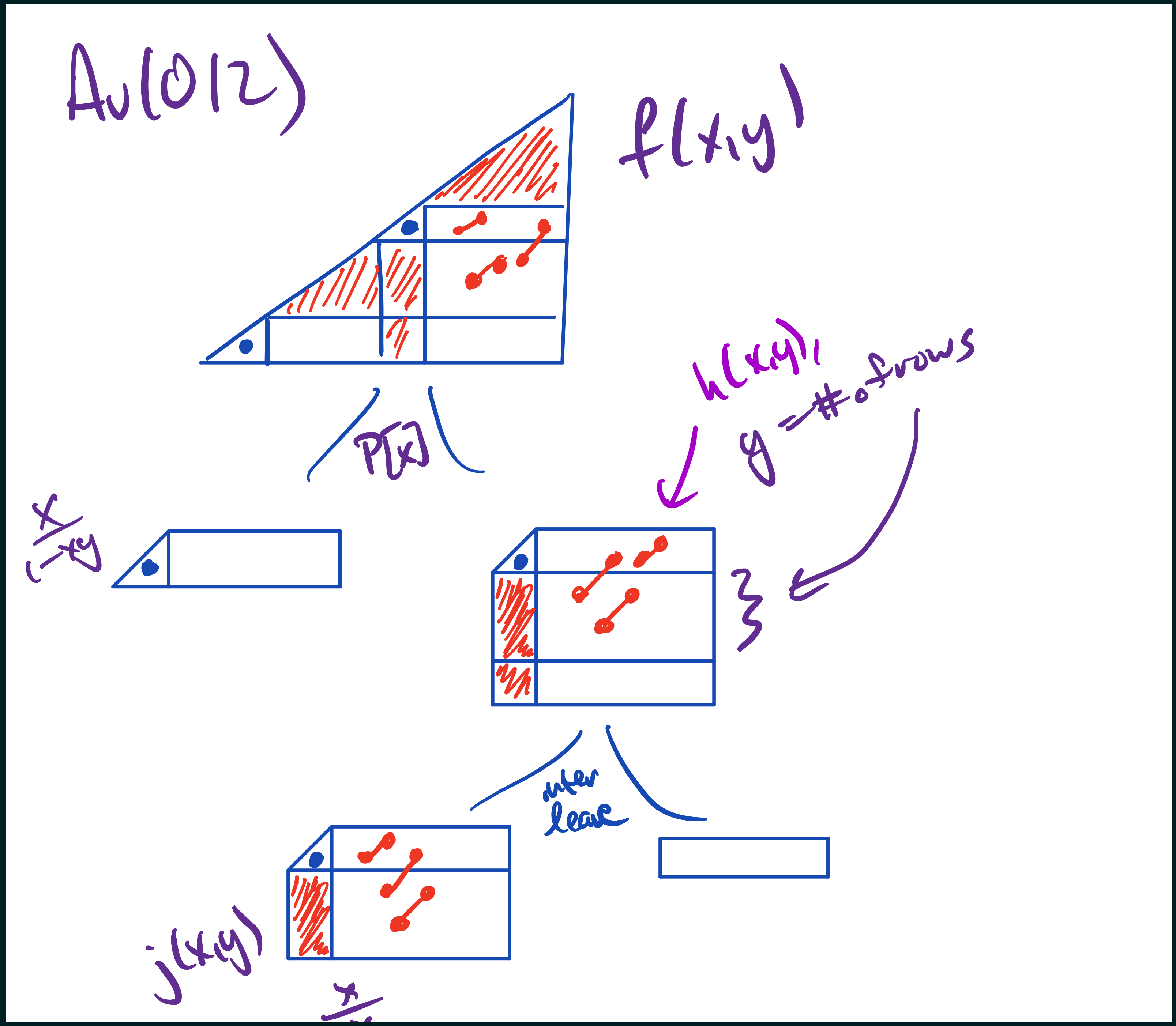
$$F = 1 + x + x \cdot G$$

$$G = x + xG$$

$$\Rightarrow G = \frac{x}{1-x}$$

$$\Rightarrow F = 1 + x + \frac{x^2}{1-x}$$

$$= \frac{1}{1-x}$$


Av(012)
f(x,y)
P[x]
h(x,y),
y = # of rows
inter leave
j(x,y)

Thank you!