Enumerative Properties of Cogrowth Series on Free Products of Finite Groups

ACA 2021 Session on Algorithmic Combinatorics

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Introduction

Word Problem

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Language of the Word Problem

$$L(G, S) := \{ \underbrace{s_1 s_2 \dots s_n}_{\text{symbol concatenation}} : n \geq 0, \ s_i \in S, \ \underbrace{s_1 \cdot s_2 \cdot \dots \cdot s_n}_{\text{group product}} = 1 \in G \}$$

Cogrowth Sequence and Series

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Cogrowth Sequence and Series

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$L(G,S) := \{s_1 s_2 \ldots s_n : n \geq 0,\ s_i \in S,\ s_1 \cdot s_2 \cdot \ldots \cdot s_n = 1 \in G\}$

- G: group generated by a finite set $S \subseteq G \setminus \{1\}$
- *cogrowth sequence* of G with respect to S:
$$\{|L(G,S) \cap S^n|\}_{n \geq 0}$$
- cogrowth series (GFs):
$F(t) \equiv F_{G;S}(t) := \sum_{n \geq 0} |L(G,S) \cap S^n|\, t^n \in \mathbb{Z}_{\geq 0}[[t]]$

Setup and Motivation

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- Continue the study of Bell and Mishna (2018), on free products of (finitely many) finite groups
- $G = G_1 * G_2 * \dots * G_d$; G_i cyclic or dihedral

Result of Muller and Schupp

Free products of finite groups have **algebraic** GFs

- ie. $Q(t, F(t)) = 0$ for some $Q(t,z) \in \mathbb{Z}[t,z] \setminus \{0\}$
- Call Q a **satisfying polynomial** of $F(t)$
- $\deg_z Q$ is minimum $\implies$ Q is a **minimal polynomial** of $F(t)$

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Main Objective

Obtain degree bounds on minimal polynomials.

Excursions on Cayley Graphs

We can visualize the problem using Cayley graphs.

- Cayley Graph of G with respect to S: $\chi(G,S) = (V,E)$
  - $V = G$, $E = \{(g, gs) : g \in G,\ s \in S\}$ (ie. directed)
  - Arcs show multiplication by elements of S
  - Walks show products on elements of S
- *Excursions* of $\chi(G,S)$:
  - Excursions are walks that start and end at $1 \in G$
  - $L(G,S) \leftrightarrow$ excursions on $\chi(G,S)$

The cogrowth sequence counts excursions on $\chi(G,S)$

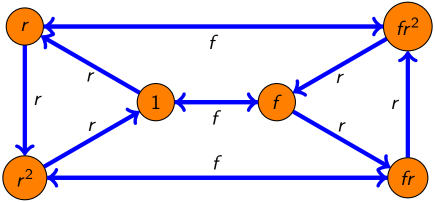
Examples: Finite Cyclic and Dihedral Groups

$G := \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n = \langle x | x^n = 1 \rangle$; $S := \{x\}$

- $L(G, S) = \{\epsilon, x^n, x^{2n}, x^{3n}, \ldots\}$
- Cogrowth GF: $F_{G;S}(t) = \frac{1}{1-t^n}$
- $\chi(G, S)$ is the directed cycle on n vertices

$G := D_n = \langle r, f | r^n = 1, f^2 = 1, rf = fr^{-1} \rangle$; $S := \{r, f\}$

- $L(D_3, S) = \{ff, rrr, rfrf, frfr, rrfrrf, rfrrfr, frrfrr\}^*$
- $\chi(D_3, S)$: See below



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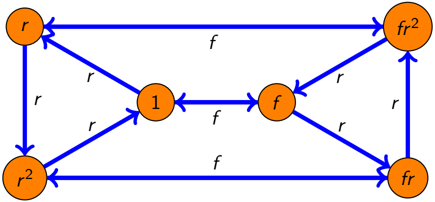
- Cogrowth GF: $F_n(t) := F_{D_n;S}(t) = ?$
- $\chi(D_3, S)$: See below



Cogrowth GF for D_n

Proposition 1

For each $n \geq 3$,

$$F_n(t) = \frac{1}{2} + \frac{1}{2n} \sum_{j=0}^{n-1} \frac{1}{1 - 2\cos(\frac{2\pi j}{n})t}. \tag{1}$$

Corollary 2

$F_n(t) = \dfrac{p(t)}{q(t)}$, *with* $p, q \in \mathbb{Z}[t]$, $p(0) = q(0) = 1$, *and* $\deg p = \deg q \leq d_n$, *where*

$$d_n := \begin{cases} \frac{n+1}{2}, & n \text{ is odd} \\ 2\lceil \frac{n}{4} \rceil, & n \text{ is even} \end{cases}. \tag{2}$$

Initial Bound on Finite Groups using Representation Theory

Lemma 3 (Bell, L., Mishna 2021+)

Let H be a finite group with degrees of irreducible representations given by $n_1, \ldots, n_d$, with T as a generating set. Let $\alpha := \sum_{s \in T} s \in \mathbb{C}[H]$, and $A(t) := \sum_{n \geq 0} \phi(\alpha^n) t^n$. Then $A(t)$ is the power series expansion of a rational function $p(t)/q(t)$ where $p, q \in \mathbb{Z}[t]$ are polynomials with $q(0) = 1$ and

$$(\deg p) + 1, \deg q \leq n_1 + \cdots + n_d \leq |H|.$$

In particular, if $\deg q = |H|$ or $\deg p = |H| - 1$, then H is abelian.

Initial Bound on Finite Groups using Representation Theory

Proof.

- Consider an isomorphism $\Psi : \overline{\mathbb{Q}}[H] \to M_{n_1}(\overline{\mathbb{Q}}) \times \cdots \times M_{n_d}(\overline{\mathbb{Q}})$.
- Ψ induces a $\overline{\mathbb{Q}}$-algebra isomorphism between the power series rings $\overline{\mathbb{Q}}[H][[t]]$ and $\left(M_{n_1}(\overline{\mathbb{Q}}) \times \cdots \times M_{n_d}(\overline{\mathbb{Q}})\right)[[t]]$ sending $\sum_{n\geq 0} \alpha^n t^n \mapsto \sum_{n\geq 0}(Y_1^n, \ldots, Y_d^n)t^n$ where $\Psi(\alpha) = (Y_1, \ldots, Y_d)$.
- By Cayley Hamilton, this image satisfy a linear recurrence of order at most $n_1 + \cdots + n_d$.
- Thus, $A(t) := \sum \alpha^n t^n = p(t)/q(t)$ with $p, q \in \overline{\mathbb{Q}}[t]$ coprime, and $q(0) = 1$.
- Since $A(t) \in \mathbb{Z}[[t]]$, p/q must be invariant under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.
- The roots of $q(t^{-1})$ are algebraic integers, so $p, q \in \mathbb{Z}[t]$.

$\square$

Initial Bound on Finite Groups using Representation Theory

Example ($G = \mathbb{Z}_n = \langle x | x^n = 1 \rangle; S = \{x\}$)

Lemma 3 implies that $F_{G;S}(t) \equiv \frac{1}{1-t^n} = \frac{p(t)}{q(t)}$, where $p, q \in \mathbb{Z}[t]$, with $\deg p \leq n-1$ and $\deg q \leq n$.

Here, $\deg p = 0$ and $\deg q = n$.

Example ($G = D_n \equiv \langle r, f | r^n{=}1, f^2{=}1, rf = fr^{-1} \rangle; S := \{r, f\}$)

- Sum, $N := n_1 + n_2 + \ldots + n_d$ is $n+2$ if n is even; $n+1$ if n is odd
- Lemma 3 $\implies$ $F_{G;S}(t) = p(t)/q(t)$, $\deg p \leq N-1$, $\deg q \leq N$.
- Corollary 2 $\implies$ $\deg p = \deg q \leq \frac{N}{2}$.

Free Products of Finite Groups

Definition

Let $G_1, G_2, \ldots, G_m$ be groups. The free product of $G_1, G_2, \ldots, G_m$, denoted as $G := G_1 * G_2 * \ldots * G_m = \coprod_{i=1}^{m} G_i$, is the group generated by $\cup_{i=1}^{m} G_i$, subject to the relations in each G_i, and the identity element in each G_i is identified with $1 \in G$.
If K is any group and $m \geq 0$, we define $K^{*m} := \underbrace{K * K * \ldots * K}_{m \text{ factors}}$.

Example

$\mathbb{Z}_2^{*3} \equiv \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \equiv \langle x, y, z | x^2 = 1, y^2 = 1, z^2 = 1 \rangle$ $(xy \neq yx)$

Important

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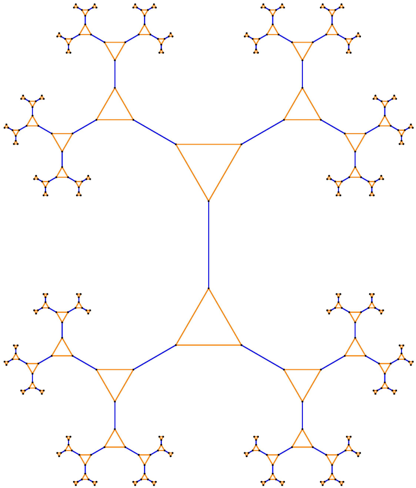
Important

The cogrowth GF of G depends on the cogrowth GF of each G_i in a nontrivial way

- We focus on the case where each G_i is finite
- Generating set: $S = \cup_{i=1}^{m} S_i$; S_i a generating set for G_i

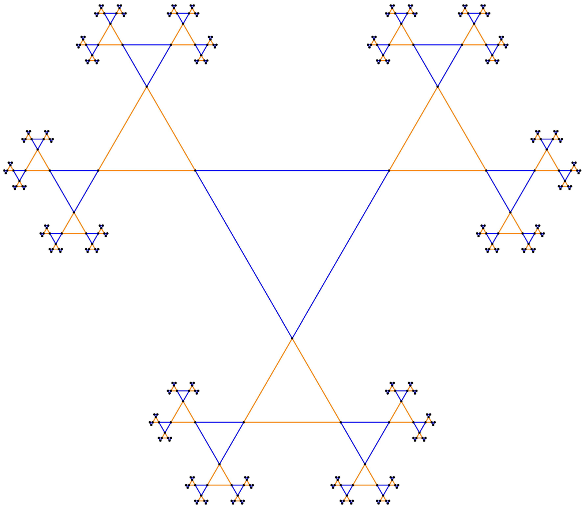
Visualization of Cayley Graphs

$$\mathbb{Z}_2 * \mathbb{Z}_3 \equiv \langle x | x^2 = 1 \rangle * \langle y | y^3 = 1 \rangle$$



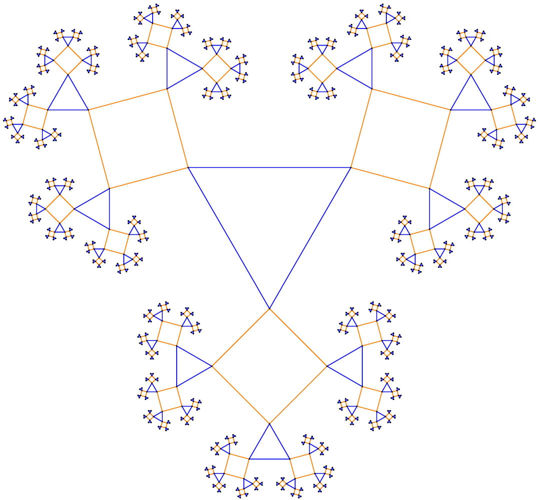
Visualization of Cayley Graphs

$$\mathbb{Z}_3 * \mathbb{Z}_3 \equiv \langle x | x^3 = 1 \rangle * \langle y | y^3 = 1 \rangle$$



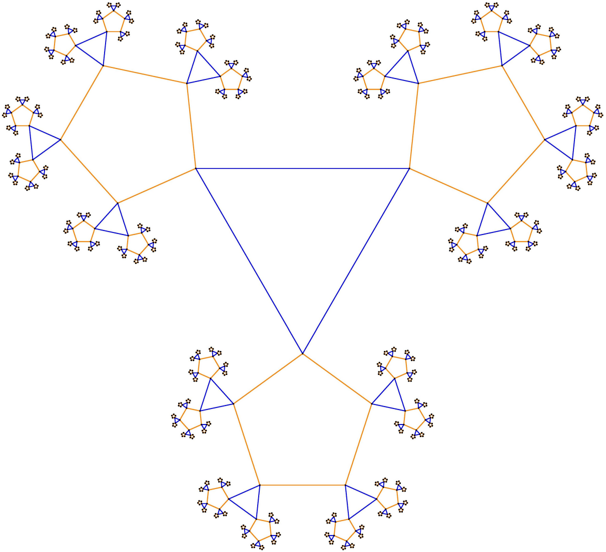
Visualization of Cayley Graphs

$$\mathbb{Z}_3 * \mathbb{Z}_4 \equiv \langle x | x^3 = 1 \rangle * \langle y | y^4 = 1 \rangle$$



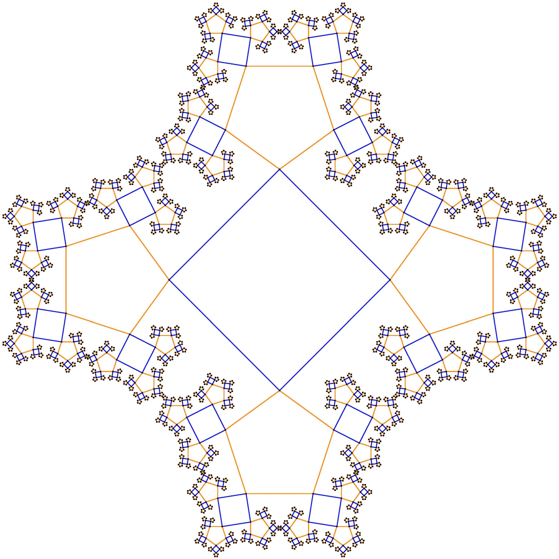
Visualization of Cayley Graphs

$$\mathbb{Z}_3 * \mathbb{Z}_5 \equiv \langle x | x^3 = 1 \rangle * \langle y | y^5 = 1 \rangle$$



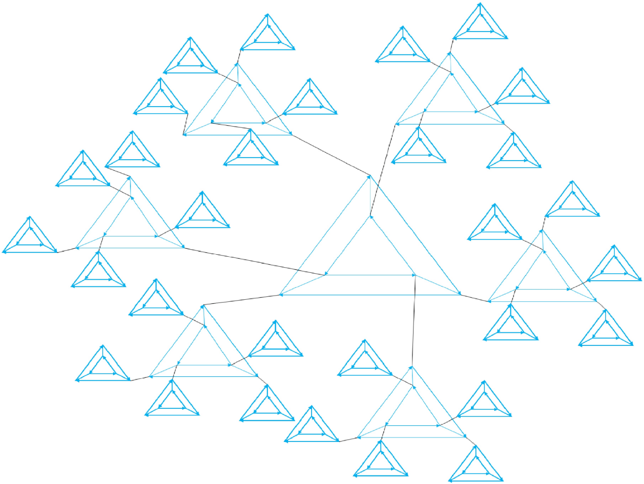
Visualization of Cayley Graphs

$$\mathbb{Z}_4 * \mathbb{Z}_5 \equiv \langle x | x^4 = 1 \rangle * \langle y | y^5 = 1 \rangle$$



Visualization of Cayley Graphs

$\mathbb{Z}_2 * D_3 \ (S = \{x, r, f\})$



System using Combinatorial Grammar

- $g \in G$, $X \subseteq G$, τ: an atom, ι: characteristic function.
- $Z_{g,X}$: (combinatorial class of) words in S^* evaluating to g, with proper nonempty prefixes avoiding X

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Lemma 4 (Bell and Mishna)

Let $G = G_1 * G_2 * \ldots * G_m$ be a (possibly trivial) free product of m finitely generated groups. Let S_i be a finite generating set for G_i so that $S = \cup_{i=1}^m S_i$ is a generating set for G. For each $1 \leq i \leq m$ and $\{g\} \cup X \subseteq G_i$, using disjoint unions of combinatorial classes,

1. $Z_{g,X} = (\iota(g \in S_i \cap X)\tau) \cup \left(\bigcup_{s \in S_i \setminus X} \left(\tau \times Z_{s^{-1}g, s^{-1}X}\right)\right)$, *if* $1 \in X,\ g \neq 1$.
2. $Z_{g,X} = Z_{1,X} \times Z_{g,X\cup\{1\}}$, *if* $1 \notin X,\ g \neq 1$.
3. $Z_{1,X} = \epsilon \cup \left(Z_{1,X} \times \left(Z_{1,X\cup\{1\}} \setminus \epsilon\right)\right)$, *if* $1 \notin X$.
4. $Z_{1,X} = \epsilon \cup \left(\bigcup_{s \in S \setminus S_i} (\tau \times Z_{s^{-1},\{s^{-1}\}})\right) \cup \left(\bigcup_{s \in S_i \setminus X} (\tau \times Z_{s^{-1}, s^{-1}X})\right)$, *if* $1 \in X$.

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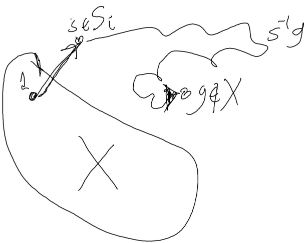
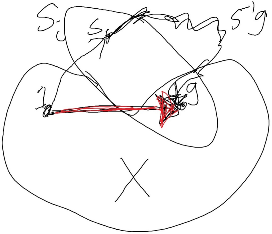
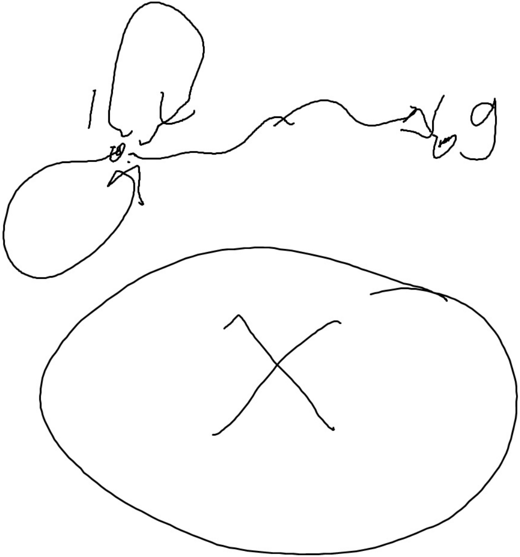




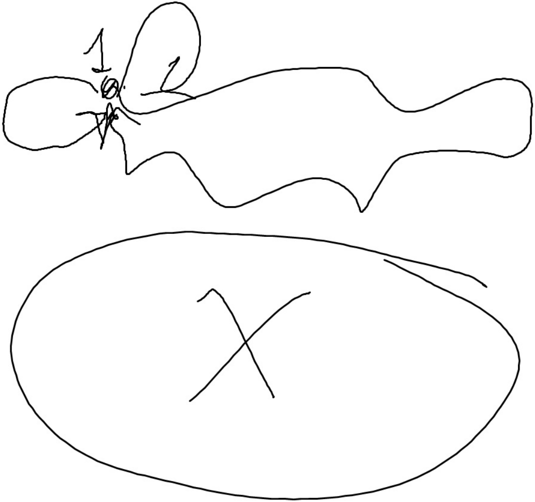
System using Combinatorial Grammar

2. $Z_{g,X} = Z_{1,X} \times Z_{g,X \cup \{1\}}$, if $1 \notin X$, $g \neq 1$.



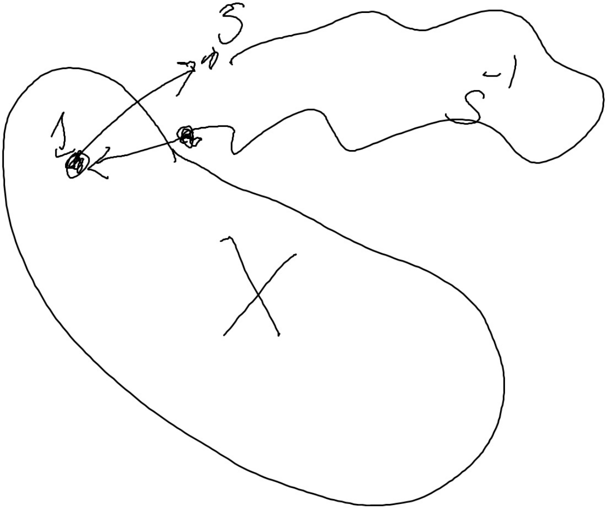
System using Combinatorial Grammar

3. $Z_{1,X} = \epsilon \cup \left(Z_{1,X} \times \left(Z_{1,X\cup\{1\}} \setminus \epsilon\right)\right)$, if $1 \notin X$.



System using Combinatorial Grammar

4. $Z_{1,X} = \epsilon \cup \left(\bigcup_{s \in S \setminus S_i} (\tau \times Z_{s^{-1},\{s^{-1}\}}) \right) \cup \left(\bigcup_{s \in S_i \setminus X} (\tau \times Z_{s^{-1},s^{-1}X}) \right)$, if $1 \in X$.



Combinatorial Grammar to GF

- $F_{g,X}$: GF for $Z_{g,X}$

Corollary 5 (Bell and Mishna)

Adopting the same notation used in Lemma 4, we have the analogous equalities for the set of generating functions $\{F_{g,X}\}$.

1. $F_{g,X}(t) = \iota(g \in S_i \cap X)t + \sum_{s \in S_i \setminus X} tF_{s^{-1}g, s^{-1}X}(t)$ *if* $1 \in X,\ g \neq 1$.
2. $F_{1,X}(t) = 1 + F_{1,X}(t)(F_{1,X\cup\{1\}}(t) - 1)$ *if* $1 \notin X$.
3. $F_{g,X}(t) = F_{1,X}(t)F_{g,X\cup\{1\}}(t)$ *if* $1 \notin X,\ g \neq 1$.
4. $F_{1,X}(t) = 1 + \sum_{s \in S \setminus S_i} tF_{s^{-1},\{s^{-1}\}}(t) + \sum_{s \in S_i \setminus X} tF_{s^{-1},s^{-1}X}(t)$ *if* $1 \in X$.

Consequences and Obstructions

- Each G_i is finite $\implies$ the combinatorial grammar contains only finitely many equations.
- We can eliminate variables on the grammar to obtain a satisfying polynomial.
- Obstructions:
  - The size of the initial system can be large (exponential in $|G_i|$)
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- Each G_i is finite $\implies$ the combinatorial grammar contains only finitely many equations.
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- Obstructions:
  - The size of the initial system can be large (exponential in $|G_i|$)
  - Elimination process is time consuming, even for a computer
- Solution: Use free probability theory and obtain a system of size linear in $|G_i|$

Free Probability: A Brief Introduction

- $G = G_1 * \ldots * G_m$; $S = \cup_{i=1}^m S_i$ as before
- Group algebra $\mathbb{C}[G]$: non-commutative random variables
- linear **expectation** operator $\phi : \mathbb{C}[G] \to \mathbb{C}$,

$$\phi\left(\left(\sum_{g \in G} \alpha_g g\right)\right) = \alpha_1$$

- $\left\{\phi\left(\left(\sum_{s \in S} s\right)^n\right)\right\}_{n \geq 0}$: cogrowth sequence
- *Cauchy transform* of $\alpha \in \mathbb{C}[G]$: $G_\alpha(t) := \sum_{n \geq 0} \phi(\alpha^i) t^{-n-1}$
- Inverse Cauchy transform of α: $K_\alpha = G_\alpha^{\langle -1 \rangle}$.
- **Important Fact**: For $\alpha = \sum \alpha_g g$, $\beta = \sum \beta_g g$, if $\alpha_g \beta_g = 0$ for each $g \in G$, then $K_{\alpha+\beta}(t) = K_\alpha(t) + K_\beta(t) - t^{-1}$.

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Upper Bound on Resultants

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Reduced Resultant

$$\overline{\operatorname{Res}}(f, g, z) := \begin{cases} \operatorname{trim} \operatorname{Res}(f, g, z), & (\deg_z f)(\deg_z g) > 0 \\ \operatorname{trim} f, & \deg_z f = 0,\ \deg_z g > 0 \\ \operatorname{trim} g, & \deg_z f > 0,\ \deg_z g = 0 \\ 1, & \deg_z f = \deg_z g = 0 \end{cases}$$

where $\operatorname{trim} f = f \cdot \prod_v v^{-\operatorname{val}_v f}$.

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where $\operatorname{trim} f = f \cdot \prod_v v^{-\operatorname{val}_v f}$.

Purpose of $\overline{\operatorname{Res}}$

To remove monomial factors and redundant exponents in order to decrease the degrees of the polynomials.

Algorithm for Algebraic Elimination

Algorithm 1 polynomial elimination over an integral domain B

Input: $n \in \mathbb{Z}_{>0}$; $t, z_1, \ldots, z_n$ indeterminate; $\vec{P} \in B[t, z_1, \ldots, z_n]^n$.
Assumption: There are algebraic functions $F_1(t), \ldots, F_n(t)$, all nonzero, such that $\vec{P}(t, F_1(t), \ldots, F_n(t)) = 0$.
Purpose: Find $P_f(t,z) \in B[t,z]$, $P_f \not\equiv 0$ so that for any sequence of nonzero algebraic functions, $F_1(t), \ldots, F_n(t)$, it holds that $\vec{P}(t, F_1(t), \ldots, F_n(t)) = 0 \implies P_f(t, F_1(t)) = 0$.

```
$\vec{P}^{(0)} := \vec{P}$
for $k = 1, 2, \ldots, n-1$ do
    for $i = 1, 2, \ldots, n-k$ do
        $P_i^{(k)} := \overline{\mathrm{Res}}_B(P_i^{(k-1)}, P_{n-k+1}^{(k-1)}, z_{n-k+1})$
    end for
    $\vec{P}^{(k)} := (P_i^{(k)})_{i=1}^{n-k}$
end for
return $P_f(t,z) := P_1^{(n-1)}(t,z) \in B[t,z]$
```

A First Bound on Free Products of Finite Groups

Theorem 6 (Bell, L., Mishna 2021+)

Let $G_1, \ldots, G_r$ be finite groups with generating sets $S_1, S_2, \ldots, S_r$ respectively. Let Δ_i denote the sum of the degrees of the irreducible representations of G_i for $i = 1, \ldots, r$. Then the cogrowth series $F(t)$ of $\coprod_{i=1}^r G_i^{*m_i}$ with respect to the generating set $S := \cup_{i=1}^r S_i$, is algebraic and satisfies $Q(t, F(t)) = 0$, where $Q(t, z) \in \mathbb{Z}[t, z]$ with $\deg_t(Q)$ and $\deg_z(Q)$ both at most

$$\left(\prod_{i=1}^r \Delta_i\right)\left(1 + \sum_{i=1}^r \frac{1}{\Delta_i}\right).$$

- Theorem 6 $\implies$ second inequality of Eqn (4) in Theorem 9
- Theorem 6 is applicable to any finite groups with any generating sets

Free Products of Cyclic Groups

- $G := \coprod_{i=1}^{r} \coprod_{j=1}^{m_i} \langle x_{ij} | x_{ij}^{n_i} = 1 \rangle = \mathbb{Z}_{n_1}^{*m_1} * \mathbb{Z}_{n_2}^{*m_2} * \ldots * \mathbb{Z}_{n_r}^{*m_r}$
- $S = \{x_{ij} : 1 \leq i \leq r, 1 \leq j \leq m_i\}$
- r: number of <u>distinct</u> cyclic factors

Free Products of Cyclic Groups

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- $S = \{x_{ij} : 1 \leq i \leq r, 1 \leq j \leq m_i\}$
- r: number of distinct cyclic factors
- Using free probability, we obtain a system of equations for $z = F_{G;S}(t)$.

Free Products of Cyclic Groups

Theorem 7 (Liu)

For $n_i \geq 2$ and $m_i \geq 1$, let

$$G := \coprod_{i=1}^{r} \coprod_{j=1}^{m_i} \langle x_{ij} | x_{ij}^{n_i} = 1 \rangle = \mathbb{Z}_{n_1}^{*m_1} * \mathbb{Z}_{n_2}^{*m_2} * \ldots * \mathbb{Z}_{n_r}^{*m_r},$$

and $S := \{x_{ij} | i = 1, \ldots, r; j = 1, \ldots, m_i\}$. Let $F(t) := F_{G;S}(t)$ be the cogrowth GF. Then the system,

$$P_i(t, z, z_1, \ldots, z_r) := tzz_i^{n_i} - z_i^{n_i - 1} - tz = 0, \;\; i = 1, \ldots, r;$$
$$P_{r+1}(t, z, z_1, \ldots, z_r) := z - (\sum_{j=1}^{r} m_j tzz_j) + (\sum_{j=1}^{r} m_j) - 1 = 0 \quad (3)$$

solves $F(t)$: There are algebraic functions $F_j(t) \not\equiv 0$, such that $P_i(t, F(t), F_1(t), \ldots, F_r(t)) = 0$ for $1 \leq i \leq r+1$.

Case of Identical Cyclic Factors($r = 1$): $G = \mathbb{Z}_n^{*m}$

Solved by Bell and Mishna using combinatorial grammar.

$$\boldsymbol{Q(t,z) = (z-1)(z+m-1)^{n-1} - m^n t^n z^n}$$

From Free Probability,

- $P_1^{(0)} = P_1(t,z,z_1) = tzz_1^{n_1} - z_1^{n_1-1} - tz$;
- $P_2^{(0)}(t,z,z_1) = P_2(t,z,z_1) = z - mtzz_1 + m - 1$;
- $P_1^{(1)}(t,z) = (z-1)(z+m-1)^{n-1} - m^n t^n z^n \boldsymbol{= Q(t,z)}$.

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- $P_1^{(1)}(t,z) = (z-1)(z+m-1)^{n-1} - m^n t^n z^n \boldsymbol{= Q(t,z)}$.

Key Observation

The degree of satisfying polynomial is independent of m.
We can generalize this result to an arbitrary number of distinct free factors.

Case of Two Distinct Factors ($r = 2$)

- $G := \mathbb{Z}_{n_1}^{*m_1} * \mathbb{Z}_{n_2}^{*m_2}$
- System of polynomials:

$$
\begin{aligned}
P_1 &= tzz_1^{n_1} - z_1^{n_1-1} - tz \\
P_2 &= tzz_2^{n_2} - z_2^{n_2-1} - tz \\
P_3 &= z - m_1 tzz_1 - m_2 tzz_2 + m_1 + m_2 - 1.
\end{aligned}
$$

- After one iteration of Algorithm 1: $P_1^{(1)} = P_1$;
$$P_2^{(1)} = \begin{cases} (z - m_1 tzz_1 + m_1 - 1)(z - m_1 tzz_1 + m - 1)^{n_2-1} - (m_2 tz)^{n_2}, & m_1 > 1 \\ (1 - tz_1)(z - tzz_1 + m_2)^{n_2-1} - m_2^{n_2} t^{n_2} z^{n_2-1}, & m_1 = 1 \end{cases}$$

Case of Two Distinct Factors ($r = 2$)

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P_2 &= tzz_2^{n_2} - z_2^{n_2-1} - tz \\
P_3 &= z - m_1 tzz_1 - m_2 tzz_2 + m_1 + m_2 - 1.
\end{aligned}
$$

- After one iteration of Algorithm 1: $P_1^{(1)} = P_1$;

$$
P_2^{(1)} = \begin{cases} (z - m_1 tzz_1 + m_1 - 1)(z - m_1 tzz_1 + m - 1)^{n_2-1} - (m_2 tz)^{n_2}, & m_1 > 1 \\ (1 - tz_1)(z - tzz_1 + m_2)^{n_2-1} - m_2^{n_2} t^{n_2} z^{n_2-1}, & m_1 = 1 \end{cases}
$$

- Upper bound on resultants: $\deg_z \mathrm{Res}(P_1^{(1)}, P_2^{(1)}, z_1) \leq n_2 + n_1(n_2 - 1)$

Degree Bound Theorem for $r = 2$

Theorem 8 (Liu)

Let

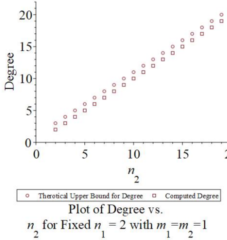
$$G = \mathbb{Z}_{n_1}^{*m_1} * \mathbb{Z}_{n_2}^{*m_2} = \coprod_{i=1}^{2} \coprod_{j=1}^{m_i} \langle x_{ij} | x_{ij}^{n_i} = 1 \rangle$$

be generated by $S = \{x_{ij} : i = 1, 2;\ 1 \leq j \leq m_i\}$. *Then there is a satisfying polynomial* $Q \in \mathbb{Z}[t, z] \setminus \{0\}$ *for the cogrowth series* $F_{G;S}(t)$ *such that* $\deg_z Q$ *satisfy the upper bounds given in the table below.*

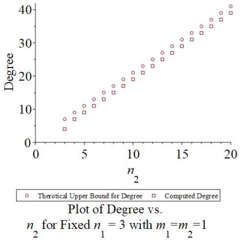
	$m_2 = 1$	$m_2 > 1$
$m_1 = 1$	$1 + n_1 n_2 - \max\{n_1, n_2\}$	$1 + n_1(n_2 - 1)$
$m_1 > 1$	$1 + n_2(n_1 - 1)$	$1 + n_1 n_2$

Upper Bounds for $\deg_z Q$ for $r = 2$ based on the values of m_1, m_2.

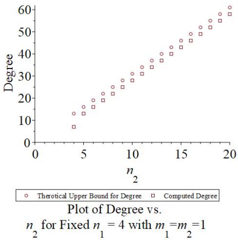
Plots for $r = 2$: Computed Degrees vs. Upper Bounds



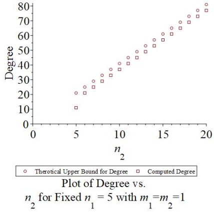
(a) $n_1 = 2$



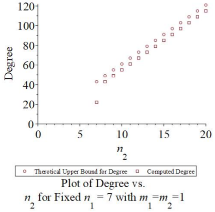
(b) $n_1 = 3$



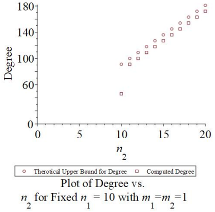
(c) $n_1 = 4$



(d) $n_1 = 5$



(e) $n_1 = 7$



(f) $n_1 = 10$

Plots of actual degrees and upper bounds vs. $n_2 = n_1, \ldots, 20$ for various fixed n_1; $m_1 = m_2 = 1$.

Conjectures for $r = 2$

	$m_2 = 1$	$m_2 > 1$
$m_1 = 1$	$1 + n_1 n_2 - \max\{n_1, n_2\}$	$1 + n_1(n_2 - 1)$
$m_1 > 1$	$1 + n_2(n_1 - 1)$	$1 + n_1 n_2$

Upper Bounds for $\deg_z Q$ for $r = 2$ based on the values of m_1, m_2.

- Entries in our table of upper bounds can be decreased precisely by one.
- If $m_1 = m_2 = 1$, then

$$deg_z Q \leq 1 + n_1 n_2 - \max\{n_1, n_2\} - \min\{n_1, n_2\} + 1 = 2 + n_1 n_2 - n_1 - n_2$$

Arbitrary Number of Distinct Cyclic Factors

Theorem 9 (Liu)

Fix $r \geq 3$. As before, consider the group,

$$G := \coprod_{i=1}^{r} \coprod_{j=1}^{m_i} \langle x_{ij} | x_{ij}^{n_i} = 1 \rangle = \mathbb{Z}_{n_1}^{*m_1} * \mathbb{Z}_{n_2}^{*m_2} * \ldots * \mathbb{Z}_{n_r}^{*m_r}$$

generated by $S := \{x_{ij}\}$, $n_i \geq 2$; $m_i \geq 1$. Running Algorithm 1 with input as the system (3), given in Theorem 7, we deduce

$$\begin{aligned} \deg_z P_1^{(r)} &\leq (n_1 n_2 \ldots n_r) \left(1 + \frac{1}{n_{r-1} n_r} + \sum_{k=1}^{r-2} \frac{1}{n_k} \right) \\ &< (n_1 n_2 \ldots n_r) \left(1 + \sum_{k=1}^{r} \frac{1}{n_k} \right), \end{aligned} \quad (4)$$

and for $0 \leq k < r$, $1 \leq j \leq r - k$,

$$\deg_{z_j} P_{r-k+1}^{(k)} \leq n_{r-k+1} \ldots n_{r-1} n_r .$$

Identical Dihedral Factors

- $G = D_n^{*m} \equiv \coprod_{i=1}^{m} \langle r_i, f_i | r_i^n = 1, f_i^2 = 1, r_i f_i = f_i r_i^{-1} \rangle$
- $S = \{r_1, f_1, r_2, f_2, \ldots, r_m, f_m\}$
- Obtaining $Q(t, z)$ explicitly in this case is difficult, since GFs for dihedral groups are not geometric series

Identical Dihedral Factors

Recall! (from Prop. 1)

$$d_m := \begin{cases} \frac{m+1}{2}, & m \text{ is odd} \\ 2\lceil \frac{m}{4} \rceil, & m \text{ is even} \end{cases}.$$

Proposition 10

Let $G = D_n^{*m} = \coprod_{i=1}^{m} \langle r_i, f_i | r_i^n = 1, f_i^2 = 1, r_i f_i = f_i r_i^{-1} \rangle$ with the generating set, $S = \{r_1, f_1, r_2, f_2, \ldots, r_m, f_m\}$. Then the cogrowth series, $F(t) := F_{G;S}(t)$, has a satisfying polynomial $P(t, z) \in \mathbb{Z}[t, z]$ with $\deg_t P \leq d_n$ and $\deg_z P \leq d_n + 1$.

Table of Degrees and Leading Coefficients: D_n^{*m}

n	d_n	$deg_z P$	$\deg_t P$	Leading coefficient in z	$P(t,0)$
3	2	3	2	$(mt+1)(2mt-1)$	$(m-1)^2$
4	2	3	2	$(4m^2t^2-1)$	$(m-1)^2$
5	3	4	3	$-(2mt-1)(m^2t^2-mt-1)$	$(m-1)^3$
6	4	5	4	$-(4m^2t^2-1)(m^2t^2-1)$	$(m-1)^4$
7	4	5	4	$-(2mt-1)(m^3t^3+2m^2t^2-mt-1)$	$(m-1)^4$
8	4	5	4	$-(4m^2t^2-1)(2m^2t^2-1)$	$(m-1)^4$
9	5	6	5	$(2mt-1)(mt+1)(m^3t^3-3m^2t^2+1)$	$(m-1)^5$
10	6	7	6	$(2mt-1)(2mt+1)O((mt)^4)$	$(m-1)^6$
11	6	7	6	$(2mt-1)O((mt)^5)$	$(m-1)^6$
12	6	7	6	$(3m^2t^2-1)(4m^2t^2-1)(m^2t^2-1)$	$(m-1)^6$

Properties of satisfying polynomials $P(t,z)$ over $\mathbb{Z}$ for the cogrowth GF of $G = D_n^{*m}$; $S = \{r_1, f_1, r_2, f_2, \ldots, r_m, f_m\}$.

Properties of the Satisfying Polynomial: D_n^{*m}

Theorem 11 (Liu)

Let $G = D_n^{*m} = \coprod_{i=1}^{m} \langle r_i, f_i | r_i^n = 1, f_i^2 = 1, r_i f_i = f_i r_i^{-1} \rangle$ with the generating set, $S = \{r_1, f_1, r_2, f_2, \ldots, r_m, f_m\}$. Then the cogrowth GF, $F(t) := F_{G;S}(t)$, has a satisfying polynomial $P(t,z) \in \mathbb{Z}[t,z]$, with leading coefficient $L(t) := [z^{\deg_z P}]P(t,z)$ and $P(t,0) \in \mathbb{Z}[m] \setminus \{0\}$, such that the following properties hold:

1. *The polynomial, $L(t) \in \mathbb{Z}[t]$, belongs to $\mathbb{Z}[mt]$;*
2. $\deg L = \deg_t P$;
3. $2mt - 1 | L(t)$;
4. *if n is even, then $P(t,z) = P(-t,z)$; and*
5. $3 | n$ *if and only if* $mt + 1 | L(t)$.

Proof of Theorem 11

- We can deduce

$$P(t,z) = \frac{1}{tz} \sum_{k=0}^{(\deg q)+1} (z+m-1)^k (mtz)^{(\deg q)+1-k} \left(([t^k]\bar{p}) - tz([t^k]\bar{q})\right) \in \mathbb{Z}[t,z]$$

  where $\bar{p}(t) := t^{\deg q} p(t^{-1})$ and $\bar{q}(t) := t^{(\deg q)+1} q(t^{-1})$.
- $P(t,0) = (m-1)^{\deg q}$.
- Property 1: $L(t) = q(mt)$.
- Property 2: $\deg L = \deg q = d_n \geq \deg_t P$ since $\bar{q}(0) = 0$.
- Property 3: $2t-1 | q(t)$ by Proposition 1.
- Property 4: $\deg q = d_n$ is even
  - $[t^k]\bar{q} = 0$ if k is even; $[t^k]\bar{p} = 0$ if k is odd;
  - decompose the summation expression for P into odd and even indices.
- Property 5: For $A \subseteq [0,1]$ finite, $f(t) = \prod_{c \in A}(1 - 2\cos(c\pi)t)$, it holds that

$$1 + t | f(t) \iff f(-1) = 0 \iff \frac{2}{3} \in A.$$

Summary: Free Product of Cyclic and Dihedral Groups

$$G = \mathbb{Z}_{n_1}^{*r_1} * \mathbb{Z}_{n_2}^{*r_2} * \ldots * \mathbb{Z}_{n_k}^{*r_k} * D_{m_1}^{*s_1} * D_{m_2}^{*s_2} * \ldots * D_{m_l}^{*s_l}$$

The dihedral factors are each generated by a rotation and a flip.

$l = 0$ (all cyclic)	$k = 1$	Bell-Mishna
	$k = 2$	Theorem 8
	$k \geq 3$	Theorem 9
$k = 0$ (all dihedral)	$l = 1, s_1 = 1$	Proposition 1
	$l = 1, s_1 > 1$	Proposition 10; Theorem 11
	$l \geq 2$	Difficult: Not yet known
$l > 0, k > 0$	?	

Identical Finite Factors

Proposition 10 generalizes to any finite group

Identical Finite Factors

Proposition 10 generalizes to any finite group

Theorem 12 (Liu)

Suppose H is finite with generating set $T \subseteq H$. Let $G := H^{*m} \simeq H_1 * H_2 * \ldots * H_m$ with each $H_i \simeq H$ via an isomorphism $\phi_i : H \to H_i$. Consider $S := \cup_{i=1}^{m} \phi_i(T)$ which generates G. Write the cogrowth GF $F_{H;T}(t) = \frac{p(t)}{q(t)}$ with $p(0)q(0) \neq 0$. Let $M := \max\{\deg p + \delta_p, \deg q + \delta_q\}$, $\delta_p := \max\{0, \deg q - 1 - \deg p\}$, $\delta_q := \max\{0, \deg p + 1 - \deg q\}$. Then $F(t) := F_{G;S}(t)$ satisfies $P(t,z) \in \mathbb{Z}[t,z] \setminus \{0\}$ such that

1. $\deg_t P \leq M + 1$ *and* $\deg_z P \leq M + 1$;
2. *if* $\delta_q > 0$, *then* $\deg_t P \leq M$;
3. *if* $\deg p + \delta_p < \deg q + \delta_q$, *then* $\deg_t P \leq M$ *and* $\deg_z P \leq M$;
4. *if* $\delta_q > 0$ *and* $\deg p + \delta_p < \deg q + \delta_q$, *then* $\deg_t P \leq M - 1$.

General Approach to Finding Satisfying Polynomials

Given a group G and a generating set S, to construct $P(t, z)$,

- Find bounds on $\deg_t P \leq d_t$ and $\deg_z P \leq d_z$.
- Generate sufficiently many first few terms of the cogrowth GF $F_{G;S}(t)$.
- Solve a linear system for the $(d_t + 1)(d_z + 1)$ undetermined coefficients that defines $P(t, z)$.

Conclusion

- Free products of finitely many finite groups have algebraic cogrowth GFs.
- Free probability provides a useful tool for bounding degrees of satisfying polynomials.
- Degrees of minimal polynomials do not (in general) depend on the number of identical free factors.

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Possible Next Steps

- Involve other classes of finite groups.
- Experiment with other generating sets.
- Bound degrees using ideal elimination and Gröbner bases.
- Obtain results on radii of convergence of the cogrowth GFs.

Thank you for listening!
Questions?

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