A Combinatorial Construction for Two Formulas In Slater's List

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ACA 2021
Session on Algorithmic Combinatorics
Jul. 25, 2021

Definition

An integer partition is an unordered finite sum of positive integers (parts) ($\lambda_1 + \lambda_2 + \cdots + \lambda_m = n$).

For the purposes of this talk, we will write parts in increasing order.

Example

$$4 + 8 + 10 = 22$$

q-Pochhammer Symbol

Definition

For $n \in \mathbb{N}$,

$$(a;q)_n = \prod_{j=1}^{n} (1 - aq^{j-1}),$$

and for $|q| < 1$

$$(a;q)_\infty = \lim_{n \to \infty} (a;q)_n = \prod_{j=1}^{\infty} (1 - aq^{j-1}).$$

(*sine qua non* of q-series)

Euler's Partition Identity

Theorem

(combinatorial version)
For $n \in \mathbb{N}$,
the number of partitions of n into distinct parts
equals the number of partitions of n into odd parts.

(q-series version)

$$\sum_{n \geq 0} \frac{q^{\binom{n+1}{2}}}{(q;q)_n} = \frac{1}{(q;q^2)_\infty}$$

Euler's Partition Theorem Example

This example is only for the *multiplicity* side.

Example

Among all partitions of 5:

$$1+1+1+1+1, \qquad 1+1+1+2, \qquad 1+2+2,$$

$$1+1+3, \qquad 2+3, \qquad 1+4, \qquad 5,$$

only three of them are into distinct parts:

$$2+3, \qquad 1+4, \qquad 5.$$

Rogers-Ramanujan Identities (one of)

Theorem

(combinatorial version)
For any $n \in \mathbb{N}$, the number of partitions of n into distinct and non-consecutive parts equals the number of partitions into parts $\equiv \pm 1 \pmod 5$.

(q-series version)

$$\sum_{n \geq 0} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}$$

This example, too, is only for the multiplicity side.

Example

Among all partitions of 9 into distinct parts:

$$2+3+4, \qquad 1+3+5, \qquad 4+5, \qquad 1+2+6,$$
$$3+6, \qquad 2+7, \qquad 1+8, \qquad 9,$$

only five of them are free of consecutive parts:

$$1+3+5, \quad 3+6, \quad 2+7, \quad 1+8, \quad 9.$$

Initial Idea

Can we start with Euler's identity,
keep track of the consecutive pairs of parts,
then eliminate them using inclusion/exclusion?

Yes (this is the rest of the talk)

Then, We will have an *alternative series*
for the Rogers-Ramanujan identities.

How is the inclusion/exlusion supposed to work?

$$1+3+4+5+7+9+11+12+14+15+16$$

The combinatorial moves and the minimal partitions

As a warmup, let's look at the series
from the series side of Euler's Partition Identity:

$$\sum_{n \geq 0} \quad q^{\binom{n+1}{2}} \quad \frac{1}{(q;q)_n}$$

The combinatorial moves and the minimal partitions

1 3 4 5 7 9 11 12 14 15 16

Generating Function for k designated rafts

Theorem

Let λ be a partition into distinct parts having exactly k designated rafts for $k \geq 1$. A generating function for such λ is

$$\sum_{m \geq 0} q^{\binom{3k+m}{2} - 3\binom{k}{2}} \begin{bmatrix} m+k-1 \\ k-1 \end{bmatrix}_{q^{-1}} \frac{1}{(q^2; q^2)_k} (-q^{3k+m+1}; q)_\infty$$

The Main Theorem

Theorem (Slater #19)

$$(-q;q)_\infty \sum_{n\geq 0} \frac{(-1)^j q^{3j^2}}{(q^2;q^2)_j(-q;q)_{2j}} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}.$$

Proof.

For $k=0$ (i.e. no designated rafts),
the generating function is $(-q;q)_\infty$.
Combine the previous theorem,
inclusion-exclusion,

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PROOF (CONT'D).

use q-Gauss' $\left(\sum_{n \geq 0} \frac{(a;q)_n (b;q)_n (c/ab)^n}{(q;q)_n (c;q)_n} = \frac{(c/a;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (c/ab;q)_\infty} \right)$
under an appropriate limit,

and the q-binomial theorem $\left(\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{n \geq 0} \frac{(a;q)_n z^n}{(q;q)_n} \right)$.

After inclusion/exclusion,
the surviving partitions are those which can have no rafts,
i.e. partitions into distinct parts with no consecutive parts.
The first Rogers-Ramanujan identity finishes the proof.

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Tribute to others

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Inserting staircases

$$\sum_{m,n,k\geq 0} \frac{q^{\binom{n+1}{2}+d\binom{n}{2}}}{(q;q)_n} \cdot \frac{(-1)^k q^{3k^2+d\binom{2k}{2}}}{(q^2;q^2)_k} \cdot \frac{(q^{2k};q)_m q^{d\binom{m}{2}}(-q)^m}{(q;q)_m} \, q^{2dnk+dnm+2dkm}$$

generates partitions into parts
that are $(2+d)$-apart for $d \geq 0$.

Unless $d = 0$, we cannot reduce the number of summations.

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Gratitude

Thank you for your attention.

Any questions?

K. Bringmann, K. Mahlburg, K. Nataraj, Distinct parts partitions without sequences, *The Electronic Journal of Combinatorics*, **22(3)**, (2015), #P3.3.

M.D. Hirschhorn, *Developments in the Theory of Partitions*, Ph.D. thesis, University of New South Wales (1979).

K. Kurşungöz, A combinatorial construction for two formulas in Slater's list. *International Journal of Number Theory*, **17**(03), 655–663 (2021).

L. J. Slater, Further Identities of the Rogers-Ramanujan Type, *Proc. London Math. Soc. Ser. 2* **54**, 147–167 (1952).

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