Constructing Minimal Telescopers for Rational Functions in Three Discrete Variables

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Joint work with Shaoshi Chen, Qing-Hu Hou,
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Outline

- Technique of creative telescoping
- New approach for trivariate rational functions

The creative telescoping problem

GIVEN $f(n,k)$, FIND $g(n,k)$ such that

$$f(n,k) = g(n,k+1) - g(n,k).$$

Then $F(n) = \sum_{k=0}^{n} f(n,k)$ satisfies

$$F(n) = \sum_{k=0}^{n} \big(g(n,k+1) - g(n,k)\big).$$

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GIVEN $f(n,k)$, FIND $c_0(n), \ldots, c_\rho(n)$ and $g(n,k)$ such that

$$c_0(n)f(n,k) + \cdots + c_\rho(n)f(n+\rho,k) \;=\; g(n,k+1) - g(n,k).$$

Then $F(n) = \sum_{k=0}^{n} f(n,k)$ satisfies

$$c_0(n)F(n) + \cdots + c_\rho(n)F(n+\rho) = \text{explicit}(n).$$

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GIVEN $f(n,k)$, FIND $c_0(n), \ldots, c_\rho(n)$ and $g(n,k)$ such that

$$\big(c_0(n) + \cdots + c_\rho(n) S_n^\rho\big)\big(f(n,k)\big) \;=\; (S_k - 1)\big(g(n,k)\big).$$

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$$c_0(n)F(n) + \cdots + c_\rho(n)F(n+\rho) = \text{explicit}(n).$$

Notation. $S_n(f(n,k)) = f(n+1,k)$ and $S_k(f(n,k)) = f(n,k+1)$.

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$$\underbrace{\left(c_0(n)+\cdots+c_\rho(n)S_n^\rho\right)}_{\text{telescoper}}\big(f(n,k)\big) \;=\; (S_k-1)\underbrace{\big(g(n,k)\big)}_{\text{certificate}}.$$

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Generations of creative telescoping algorithms

1. Elimination in operator algebras / Sister Celine's algorithm (since $\approx$ 1947)
2. Zeilberger's algorithm and its generalizations (since $\approx$ 1990)
3. The Apagodu-Zeilberger ansatz (since $\approx$ 2005)
4. The reduction-based approach (since $\approx$ 2010)

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The reduction-based approach

- Differential case:
  - Bostan, Chen, Chyzak, Li (2010): bivariate rational functions
  - Bostan, Chen, Chyzak, Li, Xin (2013): bivariate hyperexp. funs
  - Bostan, Lairez, Salvy (2013): multivariate rational functions
  - Chen, Kauers, Koutschan (2016): bivariate algebraic functions
  - Chen, van Hoeij, Kauers, Koutschan (2018): fuchsian D-finite
  - Bostan, Chyzak, Lairez, Salvy (2018): D-finite functions
  - van der Hoeven (2020): D-finite functions

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  - Chen, Hou, H., Labahn, Wang: trivariate rational functions

Double rational summations

Consider

$$\sum_{k=0}^{n}\sum_{\ell=0}^{n} f(n,k,\ell),$$

where $f \in \mathbb{C}(n,k,\ell)$ with $\mathrm{char}(\mathbb{C}) = 0$.

Double rational summations/identities

Consider

$$\sum_{k=0}^{n}\sum_{\ell=0}^{n} f(n,k,\ell) = F(n),$$

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telescoper — certificate

Example

$$\sum_{k=0}^{n}\sum_{\ell=0}^{n} \frac{2k-n}{(k+n+1)(k-2n-1)(\ell+n+1)} = 0.$$

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$$f(n+1,k,\ell) \; - \; f(n,k,\ell)$$

$$||$$

$$g(n,k+1,\ell) \; - \; g(n,k,\ell)$$

$$+$$

$$h(n,k,\ell+1) \; - \; h(n,k,\ell)$$

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$F(n+1)$

$$-\sum_{k=0}^{n+1} f(n+1,k,n+1)$$

$$\|$$

$$\sum_{\ell=0}^{n} \Big(g(n,n+1,\ell) \; - \; g(n,0,\ell) \Big) \quad -\sum_{\ell=0}^{n} f(n+1,n+1,\ell)$$

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- Creative telescoping:

$$F(n+1) \; - \; F(n)$$

$$\|$$

$$\sum_{\ell=0}^{n}\Big(g(n,n+1,\ell) \; - \; g(n,0,\ell) \; + f(n+1,n+1,\ell)\Big)$$

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$$f(n+1,n+1,n+1)$$

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$$\frac{1}{2(n+2)(2n+3)}$$

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$$F(n+1) - F(n) = 0 \quad \text{with } F(n) = \sum_{k=0}^{n}\sum_{\ell=0}^{n} f(n,k,\ell).$$

Example

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- Verification: $F(0)=0$

$$F(n)=\sum_{k=0}^{n}\sum_{\ell=0}^{n}f(n,k,\ell)=0.$$

Example

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- Creative telescoping: —key step

$$F(n+1) - F(n) = 0 \quad \text{with } F(n) = \sum_{k=0}^{n}\sum_{\ell=0}^{n} f(n,k,\ell).$$

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Univariate Abramov reduction (1975)

Let $f \in \mathbb{C}(k)$. Then $\exists\, a, b \in \mathbb{C}[k]$ such that

$$f = \underbrace{(S_k - 1)\Big(\cdots\Big)}_{\color{red}{S_k\text{-summable}}} + \frac{a}{b}$$

with

- $\deg_k(a) < \deg_k(b)$;
- $\gcd(b, S_k^m(b)) = 1$ for all $m \in \mathbb{Z} \setminus \{0\}$.

Moreover,

$$f \text{ is } S_k\text{-summable} \quad \Longleftrightarrow \quad a = 0.$$

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S_k-remainder

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Bivariate Hou-Wang reduction (2015)

Let $f \in \mathbb{C}(k,\ell)$. Then $\exists\, a_{ij} \in \mathbb{C}(k)[\ell], d_i \in \mathbb{C}[k,\ell]$ such that

$$f = \underbrace{(S_k - 1)\Big(\cdots\Big) + (S_\ell - 1)\Big(\cdots\Big)}_{\text{summable}} + \sum_{i,j} \frac{a_{ij}}{d_i^j}$$

with

- $\deg_\ell(a_{ij}) < \deg_\ell(d_i)$;
- d_i monic and irreducible over $\mathbb{C}$;
- $d_i \neq S_k^{m_1} S_\ell^{m_2}(d_{i'})$ for all $m_1, m_2 \in \mathbb{Z}$ and $i \neq i'$.

Moreover,

$$f \text{ is summable} \quad \Longleftrightarrow \quad \text{each } a_{ij}/d_i^j \text{ is summable.}$$

Individual bivariate summability (HouWang2015)

Let $j \in \mathbb{N}, a \in \mathbb{C}(k)[\ell] \setminus \{0\}, d \in \mathbb{C}[k, \ell]$ irred., $\deg_\ell(a) < \deg_\ell(d)$.

Then a/d^j is (S_k, S_ℓ)-summable iff

- $d = p(\alpha k + \beta \ell)$ for $p \in \mathbb{C}[x]$ and $\alpha, \beta \in \mathbb{Z}$ coprime;
- $\exists\, q \in \mathbb{C}(k)[\ell]$ with $\deg_\ell(q) < \deg_\ell(d)$ such that

$$a = S_k^{\beta} S_\ell^{-\alpha}(q) - q.$$

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Corollary. d is not (k, ℓ)-integer linear $\implies$ a/d^j is not summable.

Reducing individual components

$$\frac{a}{d^j} = (S_k - 1)\Big(\cdots\Big) + (S_\ell - 1)\Big(\cdots\Big) + \text{remainder???}$$

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Reducing individual components

$$\mathbb{C}(k,\ell) \xrightarrow{S_k^{\beta} S_\ell^{-\alpha}} \mathbb{C}(k,\ell)$$

$$\mathbb{C}(k,\ell) \xrightarrow[S_k]{} \mathbb{C}(k,\ell)$$

Reducing individual components

$$\begin{array}{ccccc} \ell & k & \mathbb{C}(k,\ell) & \xrightarrow{S_k^{\beta} S_\ell^{-\alpha}} & \mathbb{C}(k,\ell) \\ \Big\downarrow & \Big\downarrow & \phi_{\alpha,\beta} \Big\downarrow & & \\ \beta^{-1}\ell - \alpha k & \beta k & \mathbb{C}(k,\ell) & \xrightarrow[S_k]{} & \mathbb{C}(k,\ell) \end{array}$$

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$$\begin{array}{rccc} \phi_{\alpha,\beta}^{-1}: & \mathbb{C}(k,\ell) & \to & \mathbb{C}(k,\ell) \\ & k & \mapsto & \beta^{-1}k \\ & \ell & \mapsto & \beta\ell + \alpha k \end{array}$$

Reducing individual components

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$$
\boxed{S_k \circ \phi_{\alpha,\beta} = \phi_{\alpha,\beta} \circ S_k^{\beta} S_\ell^{-\alpha}}
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$$a = (S_k^{\beta} S_\ell^{-\alpha} - 1)\Big(\cdots\Big) + \text{remainder???}$$

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Reducing individual components

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$$\frac{a}{d^j} = (S_k - 1)\Big(\cdots\Big) + (S_\ell - 1)\Big(\cdots\Big) + \boxed{\frac{\phi_{\alpha,\beta}^{-1}(r)}{d^j}}$$

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YES

$$\boxed{(S_k - 1)\Big(\cdots\Big) + (S_\ell - 1)\Big(\cdots\Big) + \frac{\phi_{\alpha_i,\beta_i}(r_{ij})}{d_i^j}}$$

NO

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remainder

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NO

$$\frac{a_{ij}}{d_i^j}$$

$$\Downarrow$$

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r: remainder

Theorem. f is summable $\iff$ $r = 0$.

Telescoping via reduction

GIVEN $f \in \mathbb{C}(n, k, \ell)$.

FIND $c_0, \ldots, c_\rho \in \mathbb{C}[n]$ and $g, h \in \mathbb{C}(n, k, \ell)$ such that

$$(c_0(n) + \cdots + c_\rho(n) S_n^\rho)(f) = (S_k - 1)(g) + (S_\ell - 1)(h).$$

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Existence of telescopers (ChenHouLabahnWang2016)

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A telescoper $c_0(n) + \cdots + c_\rho(n) S_n^\rho$

Remarks.

- The first linear dependency leads to a minimal telescoper.
- One can leave the certificate as an un-normalized sum.

Example (continue)

Recall

$$f(n, k, \ell) = \frac{2k - n}{(k + n + 1)(k - 2n - 1)(\ell + n + 1)}.$$

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∃ a telescoper of order ≥ 1!

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$$= 0$$

Example (continue)

Recall

$$f(n,k,\ell) = \frac{2k-n}{(k+n+1)(k-2n-1)(\ell+n+1)}.$$

$$(-1) \cdot \frac{(2k-n)/((k+n+1)(k-2n-1))}{\ell+n+1}$$

$$+ \quad 1 \cdot \frac{(2k-n)/((k+n+1)(k-2n-1))}{\ell+n+1}$$

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Recall

$$f(n,k,\ell) = \frac{2k-n}{(k+n+1)(k-2n-1)(\ell+n+1)}.$$

Therefore,

- a minimal telescoper for f is

$$L = 1 \cdot S_n + (-1);$$

Example (continue)

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Therefore,

- a minimal telescoper for f is

$$L = 1 \cdot S_n + (-1);$$

- a corresponding certificate is

$$1 \cdot (g_1, h_1) + (-1) \cdot (g_0, h_0)$$

$$= \left(-\frac{k^2+(2n+2)k-8n^2-19n-11}{(k+n+1)(k-2n-2)(k-2n-3)(\ell+n+1)}, \frac{2k-n-1}{(k+n+2)(k-2n-3)(\ell+n+1)}\right).$$

Timing (in seconds)

Test suite:

$$f(n, k, \ell) = \frac{a(n, k, \ell)}{d(n, k, \ell) \cdot d(n + \xi, k, \ell)}$$

with

- $d = P_1(\xi k - \zeta n, \xi\ell + \zeta n) \cdot P_2(\zeta n + \xi k + 2\xi\ell)$,
- $\deg(a) = m$, $\deg(P_1) = \deg(P_2) = n$, $\xi, \zeta \in \mathbb{Z}$.

(m, n, ξ, ζ)	RCT+cert	RCT	order
(1, 1, 1, 1)	0.196	0.098	1
(1, 1, 1, 5)	7.319	0.112	1
(1, 1, 1, 9)	105.548	0.123	1
(1, 1, 1, 3)	0.574	0.098	1
(1, 2, 1, 3)	17.812	0.258	1
(1, 3, 1, 3)	266.206	2.008	1
(1, 4, 1, 3)	2838.827	37.052	1
(3, 2, 1, 3)	710.810	0.480	3
(3, 2, 2, 3)	1314.809	0.751	6
(3, 2, 4, 3)	1558.440	1.528	12

Benchmark

HolonomicFunctions. A Mathematica package by Koutschan.

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- CreativeTelescoping: Chyzak's algorithm (2000)
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Example.

$$f(n,k,\ell) = \frac{4n-3}{(20n-5k-\ell-3)(20n-5k-\ell+17)(5n+k+2\ell+3)(5n+k+2\ell+8)}.$$

	Timing
RCT + cert	$\approx$ 30s
CreativeTelescoping	$\approx$ 3min
FindCreativeTelescoping	-

Summary

- Results.
  - A new bivariate reduction for rational functions
  - A new approach to trivariate rational creative telescoping

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- Future work.
  - Handle four or more variables
  - Handle trivariate hypergeometric terms