Separability Problems in Creative Telescoping

Shaoshi Chen

KLMM, AMSS
Chinese Academy of Sciences

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Algorithmic Combinatorics

Joint with Ruyong Feng, Pingchuan Ma, and Michael F. Singer

Separability problem

Problem. Decide whether $f(t, \mathbf{x})$ with $\mathbf{x} = \{x_1, \ldots, x_n\}$ satisfies

$$L(t, \partial_t)(f) = 0, \qquad \text{where } L \in \mathbb{F}(t)\langle \partial_t \rangle \setminus \{0\} \text{ with } \mathrm{char}(\mathbb{F}) = 0.$$

If L exists, say f is ∂_t-separable.

Example. $f = \sqrt{t(x^2+1)} + 1$ is D_t-separable since

$$L(f) = 0, \quad \text{where } L = 2t \cdot D_t^2 + D_t.$$

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Applications.

- Separation of variables in PDE:

$$\frac{\partial y}{\partial t} - c\frac{\partial^2 y}{\partial x^2} = 0 \quad \rightsquigarrow \quad \frac{\partial y}{\partial t} - \lambda y = 0 \quad \text{and} \quad c\frac{\partial^2 y}{\partial x^2} - \lambda y = 0.$$

- Picard–Fuchs equations for differential forms;
- Holonomic polynomial sequences;
- Creative telescoping.

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- Picard–Fuchs equations for differential forms;
- Holonomic polynomial sequences;
- Creative telescoping.

Creative telescoping

Problem. Given $f(t, x_1, \ldots, x_n) \in \mathfrak{F}$, find $L \in \mathbb{F}(t)\langle \partial_t \rangle \setminus \{0\}$ s.t.

$$L(t, \partial_t)(f) = \sum_{i=1}^{n} \partial_{x_i}(g_i), \quad \text{where } g_i \in \mathfrak{F}.$$

If L exists, call L a telescoper of type $(\partial_t, \partial_{x_1}, \ldots, \partial_{x_n})$ for f.

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Example. Let $f(n,k) = \binom{n}{k}^2$. Then $L(n, S_n)(f) = \Delta_k(g)$ with

$$L = (n+1)S_n - 4n - 2 \quad \text{and} \quad g = \frac{(2k-3n-3)k^2\binom{n}{k}^2}{(k-n-1)^2}.$$

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The WZ method uses (L, g) to prove

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Existence problem of telescopers

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1990: Telescopers exist for holonomic functions

Doron Zeilberger. A holonomic systems approach to special functions identities. *Journal of Computational and Applied Mathematics.*, 32: 321–368, 1990.

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1992: Telescopers exist for proper hypergeometric terms

Herbert S. Wilf, Doron Zeilberger. An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities. *Inventiones Mathematicae*, 108: 575–633, 1992.

Existence problem of telescopers

Problem. Decide whether $f(t, x_1, \ldots, x_n) \in \mathfrak{F}$ has a telescoper of type $(\partial_t, \partial_{x_1}, \ldots, \partial_{x_n})$?

2002: The bivariate rational discrete case

Sergei A. Abramov, Ha Q.Le. A criterion for the applicability of Zeilberger's algorithm to rational functions. *Discrete Mathematics*, 259: 1–17, 2002.

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2003: The bivariate hypergeometric case

Sergei A. Abramov. When does Zeilberger's algorithm succeed? *Advances in Applied Mathematics*, 30: 424–441, 2003.

Existence problem of telescopers

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2005: The bivariate q-hypergeometric case

William Y. C.Chen, Qing-Hu Hou and Yan-Ping Mu. Applicability of the q-analogue of Zeilberger's algorithm. *Journal of Symbolic Computation*, 39: 155–170, 2005.

Existence problem of telescopers

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2012: All of 9 bivariate rational cases

Shaoshi Chen, and Michael F. Singer. Residues and telescopers for bivariate rational functions. *Advances in Applied Mathematics*, 49: 111–133, 2012.

Existence problem of telescopers

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2015: 6 bivariate mixed hypergeometric cases

Shaoshi Chen, Frédéric Chyzak, Ruyong Feng, Guofeng Fu and Ziming Li. On the existence of telescopers for mixed hypergeometric terms. *Journal of Symbolic Computation*, 68: 1–26, 2015.

Existence problem of telescopers

Problem. Decide whether $f(t, x_1, \ldots, x_n) \in \mathfrak{F}$ has a telescoper of type $(\partial_t, \partial_{x_1}, \ldots, \partial_{x_n})$?

2016: Trivariate rational discrete case

Shaoshi Chen, Qing-Hu Hou, and George Labahn and Rong-Hua Wang. Existence problem of telescopers: beyond the bivariate case. *ISSAC '16*, 167–174, 2016.

$$L(x, S_x)(f) = \Delta_y(g) + \Delta_z(h)$$

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2019: 4 trivariate rational mixed cases

Shaoshi Chen, Lixin Du and Chaochao Zhu. Existence problem of telescopers for rational functions in three variables: the mixed cases. *ISSAC'19*, 82–89, 2019.

Existence problem of telescopers

Problem. Decide whether $f(t, x_1, \ldots, x_n) \in \mathfrak{F}$ has a telescoper of type $(\partial_t, \partial_{x_1}, \ldots, \partial_{x_n})$?

2020: All 18 trivariate rational cases

Shaoshi Chen, Lixin Du, Ronghua Wang and Chaochao Zhu. On the existence of telescopers for rational functions in three variables. *Journal of Symbolic Computation*, 104: 494–522, 2021.

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Only one case is not solved!

$$L(t, D_t)(f) = \Delta_x(g) + D_y(h) \quad \Rightarrow \quad \text{SP on algebraic functions}$$

Separability criteria: rational case

Definition. $f(t,\mathbf{x})$ is split if $f = a(t)\cdot b(\mathbf{x})$.

Theorem. Let $f = P/Q$ with $P,Q \in \mathbb{F}[t,\mathbf{x}]$ and $\gcd(P,Q) = 1$. Then

$$f \text{ is } \partial_t\text{-separable}$$

$$\Updownarrow$$

$$Q \text{ is split}$$

$$\Updownarrow$$

$$f = \sum_{i=1}^{m} a_i(t)\cdot b_i(\mathbf{x}), \qquad a_i \in \mathbb{F}(t) \text{ and } b_i \in \mathbb{F}(\mathbf{x}).$$

Hypergeometric terms

Definition. A term $H(t,\mathbf{x})$ is hypergeometric over $\mathbb{F}(t,\mathbf{x})$ if

$$\frac{S_t(H)}{H}, \quad \frac{S_{x_1}(H)}{H}, \quad \ldots, \quad \frac{S_{x_n}(H)}{H} \in \mathbb{F}(t,\mathbf{x}).$$

Gosper form. For $f \in \mathbb{F}(\mathbf{x})(t)$, $\exists z \in \mathbb{F}(\mathbf{x})$, monic $a,b,c \in \mathbb{F}(\mathbf{x})[t]$ s.t.

$$f = z \cdot \frac{S_t(c)}{c} \cdot \frac{a}{b},$$

where

- $\gcd(a, S_t^i(b)) = 1$ for all $i \in \mathbb{N}$;
- $\gcd(a,c) = 1$;
- $\gcd(b, S_t(c)) = 1$.

Separability criteria: hypergeometric case

Theorem (Petkovšek1992). Let $H(t, \mathbf{x})$ be hypergeometric with

$$\frac{S_t(H)}{H} = z \cdot \frac{S_t(c)}{c} \cdot \frac{a}{b}, \qquad \text{(Gosper from)}$$

If $e_0 H + \cdots + e_d S_t^d(H) = 0$ with $e_i \in \mathbb{F}[t]$, then

$$z \in \overline{\mathbb{F}}, \quad a \mid e_0, \quad b \mid S_t^{1-d}(e_d).$$

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Theorem (LeLi2004). Let H be hypergeom. over $\mathbb{F}(t,\mathbf{x})$. Then

H is S_t-separable

$$\Updownarrow$$

$$\frac{S_t(H)}{H} = \frac{S_t(P)}{P} \cdot r, \quad P \in \mathbb{F}(\mathbf{x})[t] \text{ and } r \in \mathbb{F}(t).$$

$$\Updownarrow$$

$$H = \sum_{i=1}^{m} a_i(t) \cdot b_i(\mathbf{x}), \qquad a_i,\ b_i \text{ hypergeom. resp..}$$

Hyperexponential functions

Definition. A term $H(t,\mathbf{x})$ is hyperexponential over $\mathbb{F}(t,\mathbf{x})$ if

$$\frac{D_t(H)}{H}, \quad \frac{D_{x_1}(H)}{H}, \quad \ldots, \quad \frac{D_{x_n}(H)}{H} \in \mathbb{F}(t,\mathbf{x}).$$

Differential Gosper form. For $f \in \mathbb{F}(\mathbf{x})(t)$, $\exists a,b,c \in \mathbb{F}(\mathbf{x})[t]$ s.t.

$$f = \frac{c'}{c} + \frac{a}{b},$$

where $\gcd(b,c) = 1$ and

$$\gcd(b, a - i\,b') = 1 \quad \text{for all } i \in \mathbb{N}.$$

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$$\Updownarrow$$

The residues of a/b at simple points are not positive integer.

Riccati equation

Let $D_t(H) = u \cdot H$ with $u \in \mathbb{F}(t, \mathbf{x})$ and

$$D_t^i(H) = P_i(u, D_t(u), \ldots, D_t^{i-1}(u)) \cdot H,$$

where P_i are polynomials s.t. $P_0 = 1$ and $P_i = D_t(P_{i-1}) + u P_{i-1}$.

Definition. For $L = e_d D_t^d + \cdots + e_0 \in \mathbb{F}(t)\langle D_t \rangle$, call

$$R(u) := \sum_{i=0}^{d} e_i \cdot P_i(u, D_t(u), \ldots, D_t^{i-1}(u)) = 0$$

the Riccati equation associated with L.

Riccati equation

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the Riccati equation associated with L.

Example. Let $L := e_2 \cdot D_t^2 + e_1 \cdot D_t + e_0$. Then

$$R(u) := e_2(D_t(u) + u^2) + e_1 \cdot u + e_0 = 0.$$

Hyperexponential solutions of LDEs

Theorem (Bronstein1992). Let $H(t,\mathbf{x})$ be hyperexp. with

$$\frac{D_t(H)}{H} = \frac{D_t(c)}{c} + \frac{a}{b}, \qquad \text{(Differential Gosper from)}$$

If $e_0 H + \cdots + e_d D_t^d(H) = 0$ with $e_i \in \mathbb{F}[t]$, then $a, b \in \mathbb{F}[t]$.

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If $e_0 H + \cdots + e_d D_t^d(H) = 0$ with $e_i \in \mathbb{F}[t]$, then $a, b \in \mathbb{F}[t]$.

Proof. Write

$$\frac{a}{b} = \sum_{i=0}^{m} \lambda_i t^i + \sum_{j=1}^{s} \sum_{\ell=1}^{m_j} \frac{\mu_{j\ell}}{(t-\alpha_j)^\ell}.$$

Claim. α_j are zeros of $e_d \in \mathbb{F}[t] \quad \Rightarrow \quad \alpha_j \in \overline{\mathbb{F}}$

$$\frac{a}{b} = \frac{\mu}{(t-\alpha)^k} + \text{HTs} \quad \Rightarrow \quad P_d = \begin{cases} \frac{\prod_{i=0}^{d-1}(\mu-i)}{(t-\alpha)^d} + \text{HTs}, & k = 1; \\ \frac{\mu^d}{(t-\alpha)^{k\cdot d}} + \text{HTs}, & k > 1. \end{cases}$$

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Proof. Write

$$\frac{a}{b} = \sum_{i=0}^{m} \lambda_i t^i + \sum_{j=1}^{s} \sum_{\ell=1}^{m_j} \frac{\mu_{j\ell}}{(t-\alpha_j)^\ell}.$$

Claim. $\lambda_i,\ \mu_{j\ell} \in \overline{\mathbb{F}}$. Let $d_m := \max_i\{im + \deg_t(e_i)\}$.

$$u = \lambda_m t^m + \text{LTs} \quad \Rightarrow \quad R(u) = \left(\sum_{i \in I} \mathrm{lc}(e_i)\lambda_m^i\right) t^{d_m} + \text{LTs}.$$

$$I := \{i \mid 0 \le i \le d,\ e_i \neq 0,\ im + \deg_t(e_i) = d_m\}.$$

Separability criteria: hyperexponential case

Theorem (LeLi2004). Let H be hyperexp. over $\mathbb{F}(t, \mathbf{x})$. Then

H is D_t-separable

$$\Updownarrow$$

$$\frac{D_t(H)}{H} = \frac{D_t(P)}{P} + r, \quad P \in \mathbb{F}(\mathbf{x})[t] \text{ and } r \in \mathbb{F}(t).$$

$$\Updownarrow$$

$$H = \sum_{i=1}^{m} a_i(t) \cdot b_i(\mathbf{x}), \qquad a_i, b_i \text{ hyperexp. resp..}$$

Existence problem: trivariate rational case

Problem. Given $f \in \mathbb{F}(t, x, y)$, decide whether $\exists\ L \in \mathbb{F}(t)\langle D_t \rangle \setminus \{0\}$ s.t.

$$L(t, D_t)(f) = \Delta_x(g) + D_y(h), \quad \text{for } g, h \in \mathbb{F}(t, x, y).$$

1 Additive decomposition:

$$f = \Delta_x(u) + D_y(v) + \sum_{i=1}^{n} \frac{\alpha_i}{y - \beta_i}, \quad \alpha_i, \beta_i \in \overline{\mathbb{F}(t, x)}.$$

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1 Additive decomposition:

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2 Theorem (ChenDuZhu2019).

f has a telescoper of type (D_t, Δ_x, D_y)

$$\Updownarrow$$

either $\alpha_i \in \overline{\mathbb{F}(t,x)}$ is D_t-separable

or $\beta_i \in \overline{\mathbb{F}(t)}$ and α_i has a telescoper of type (D_t, Δ_x).

Separability problem: bivariate algebraic case

Problem. Given $f \in \overline{\mathbb{F}(t,x)}$, decide whether $\exists\, L \in \mathbb{F}(t)\langle D_t \rangle \setminus \{0\}$ s.t.

$$L(t, D_t)(f) = 0.$$

Lemma. If $f, g \in \overline{\mathbb{F}(t,x)}$ are conjugate, then

$$f \text{ is } D_t\text{-separable} \quad \Leftrightarrow \quad g \text{ is } D_t\text{-separable.}$$

Definition. The discriminant of $\{\beta_1, \ldots, \beta_n\}$ of a finite separable extension E/F is

$$\mathrm{disc}(\{\beta_1, \ldots, \beta_n\}) := \det\left((\mathrm{Tr}_{E/F}(\beta_i \cdot \beta_j))_{1 \le i \le j \le n}\right),$$

where $\mathrm{Tr}_{E/F} : E \to F$ is the trace map.

Separability criteria

Theorem. Let $f \in \overline{\mathbb{F}(t,x)}$ with the minimal polynomial

$$P(t,x,Y) = \sum_{i=0}^{d} A_i Y^i \in \mathbb{F}[t,x,Y].$$

Then f is D_t-separable iff

- $A_d = p(x) \cdot q(t)$ with $p \in \mathbb{F}[x]$ and $q \in \mathbb{F}[t]$;
- $\exists\ \alpha \in \overline{\mathbb{F}(x)}$, $\beta \in \overline{\mathbb{F}(t)}$ s.t. $\mathbb{F}(t,x,\alpha,f) = \mathbb{F}(t,x,\alpha,\beta)$ and

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$$f = \frac{1}{q(t) \cdot D(t)} \sum_{i=0}^{\ell-1} a_i \cdot \beta^i,$$

where $a_i \in \mathbb{F}(x,\alpha)[t]$ and $D = \mathrm{disc}(\{1,\beta,\ldots,\beta^{\ell-1}\}) \in \mathbb{F}(t)$.

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$$f = \frac{1}{q(t)\cdot D(t)} \sum_{i=0}^{\ell-1} a_i \cdot \beta^i.$$

$$\Updownarrow$$

$$f = \sum_{i=1}^{m} \alpha_i(x)\cdot\beta_i(t), \quad \text{where } \alpha_i \in \overline{\mathbb{F}(x)} \text{ and } \beta_i \in \overline{\mathbb{F}(t)}.$$

Finding $\alpha(x)$ and $\beta(t)$

Let $f \in \overline{\mathbb{F}(t,x)}$ with the minimal polynomial

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Finding $\alpha(x)$: Choose $(a,\alpha) \in \mathbb{F} \times \overline{\mathbb{F}(x)}$ be s.t.

$$A_d(a,x) \neq 0, \quad P(a,x,\alpha) = 0, \quad \frac{\partial P}{\partial Y}(a,x,\alpha) \neq 0.$$

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Finding $\beta(t)$: Let $Q := \text{minpoly}(\alpha) \in \mathbb{F}(x)[Y]$ and $K = \mathbb{F}(x,\alpha)$.

Finding $\alpha(x)$ and $\beta(t)$

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Finding $\beta(t)$: Let $Q := \text{minpoly}(\alpha) \in \mathbb{F}(x)[Y]$ and $K = \mathbb{F}(x,\alpha)$.

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Let $f \in \overline{\mathbb{F}(t,x)}$ with the minimal polynomial

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- Choose $(c,b) \in \mathbb{F}^2$ s.t. $Q(c,b) = 0$, $D(t,c) \cdot \text{lc}(\bar{P})(c) \neq 0$, where

$$D(t,x) = \text{disc}(\{\alpha^i f^j \mid 0 \leq i \leq \deg_Y(Q), 0 \leq j \leq \deg_Y(\bar{P})\});$$

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- Compute a zero $\beta(t)$ of $\bar{P}(t,c,Y) \in \overline{\mathbb{F}}[t,Y]$.

Verifying the separability criterion

Assume $\mathbb{F}(t,x,\alpha,f) = \mathbb{F}(t,x,\alpha,\beta)$ and

$$f = \frac{1}{q(t)\cdot D(t)} \sum_{i=0}^{\ell-1} a_i \beta^i.$$

Let $Y = (1, f, \ldots, f^{\ell-1})$ and $Z = (1, \beta, \ldots, \beta^{\ell-1})$. Then

$D_t(Y) = A \cdot Y$ and $D_t(Z) = B \cdot Z$ with $A \in \mathbb{F}(x,\alpha)(t)^{\ell\times\ell}$, $B \in \mathbb{F}(t)^{\ell\times\ell}$.

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Theorem. f is D_t-separable $\Leftrightarrow$ $\exists$ invertible $G \in \mathbb{F}(x,\alpha)[t]^{\ell\times\ell}$ s.t.

$$D_t(G) - A\cdot G = G\cdot H,$$

where $H = \frac{D_t(q^{\ell-1}D)}{q^{\ell-1}D}\cdot I_\ell - B \in \mathbb{F}(t)^{\ell\times\ell}$.

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- Finding $\alpha \in \overline{\mathbb{F}(x)}$ and $\beta \in \overline{\mathbb{C}(t)}$:
  - Choose $(a,\alpha) = (1,x)$ with

    $$P(1,x,x) = 0 \quad \text{and} \quad \frac{\partial P}{\partial Y}(1,x,x) = -2 \neq 0.$$

    Set $K = \mathbb{C}(x,\alpha) = \mathbb{C}(x)$.
  - Since P is irreducible over K, take $\bar{P} = P$;
  - Set $D(t,x) := \mathrm{disc}(\{1,f\}) = 4t$, $B_2 = 1$, and $Q := Y - x$. Then $(0,0)$ satisfies $D(t,0)B_2(0) \neq 0$ and $Q(0,0) = 0$.
  - Set $\beta = \sqrt{t} + 1$, a zero of $P(t,0,Y) = Y^2 - 2Y + 1 - t$.

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- Finding $G \in \mathbb{C}(x)[t]^{2\times 2}$:
  - Set $D(t) := \mathrm{disc}(\{1,\beta\}) = 4t$ and

$$A = \begin{pmatrix} 0 & 0 \\ \frac{x}{2} - \frac{1}{2t} & \frac{1}{2t} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ -\frac{1}{2t} & \frac{1}{2t} \end{pmatrix}.$$

  - Set $Z = (z_{ij}) \in \mathbb{C}(x)[t]^{2\times 2}$ and the system

$$D_t(Z) = AZ - Z(B - 1/t \cdot I_2)$$

    has a solution basis

$$\left\{ Q_1 := \begin{pmatrix} t & 0 \\ xt^2 + t & 0 \end{pmatrix}, \, Q_2 := \begin{pmatrix} 0 & 0 \\ -t & t \end{pmatrix} \right\}.$$

  - Since $\det(c_1Q_1 + c_2Q_2) = c_1c_2t^2 \neq 0$, y is D_t-separable.

Summary

Separability Problem. Decide whether $f(t, x_1, \ldots, x_n)$ satisfies

$$L(t, \partial_t)(f) = 0, \qquad \text{where } L \in \mathbb{F}(t)\langle \partial_t \rangle \setminus \{0\}.$$

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Thank you!