Rational Ehrhart Theory

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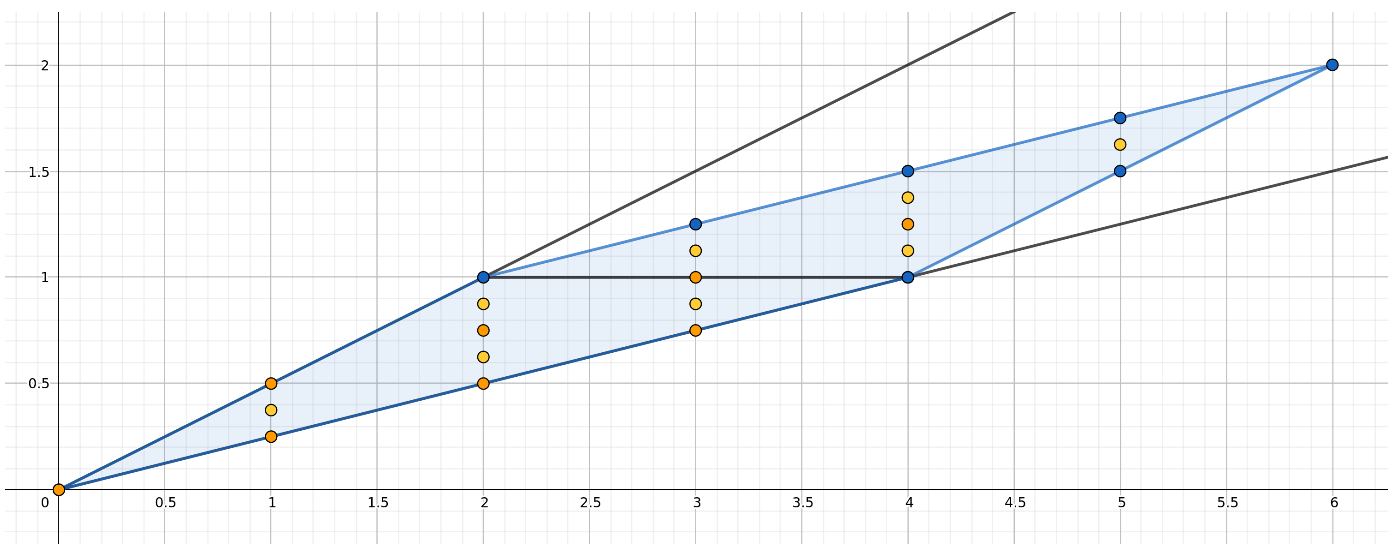
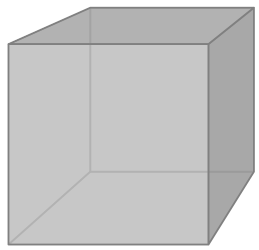


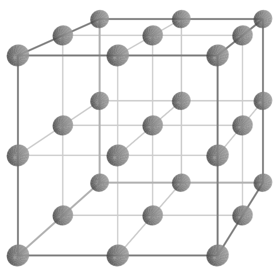
Measuring Polytopes

Rational polytope — convex hull of finitely many points in $\mathbb{Q}^d$
— solution set of a system of linear (in-)equalities with integer coefficients

Goal: measuring...



- volume $\mathrm{vol}(\mathcal{P}) = \lim_{n\to\infty} \frac{1}{n^d} \left| \mathcal{P} \cap \frac{1}{n}\mathbb{Z}^d \right|$



- discrete volume $\left| \mathcal{P} \cap \mathbb{Z}^d \right|$

Ehrhart function $\mathrm{ehr}_{\mathcal{P}}(n) := \left| \mathcal{P} \cap \frac{1}{n}\mathbb{Z}^d \right| = \left| n\mathcal{P} \cap \mathbb{Z}^d \right|$ for $n \in \mathbb{Z}_{>0}$

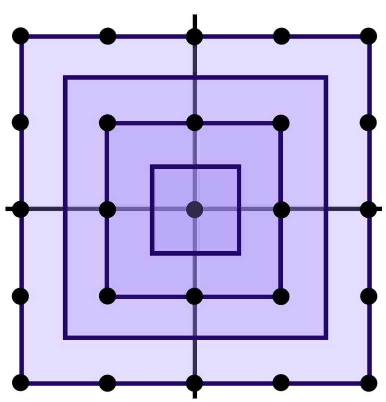
Discrete Volumes & Ehrhart Quasipolynomials

Rational polytope — convex hull of finitely many points in $\mathbb{Q}^d$

$q(n) = c_d(n)\, n^d + \cdots + c_0(n)$ is a quasipolynomial if $c_0(n), \ldots, c_d(n)$ are periodic functions; the lcm of their periods is the period of $q(n)$.

Theorem (Ehrhart 1962) For any rational polytope $\mathcal{P} \subset \mathbb{R}^d$, $\mathrm{ehr}_{\mathcal{P}}(n) := \left| n\mathcal{P} \cap \mathbb{Z}^d \right|$ is a quasipolynomial in the integer variable n whose period divides the lcm of the denominators of the vertex coordinates of P (the denominator of P).

Example $\mathcal{P} = [-\frac{1}{2}, \frac{1}{2}]^2$



$$\mathrm{ehr}_{\mathcal{P}}(n) = \begin{cases} (t+1)^2 \text{ if } t \text{ is even,} \\ t^2 \text{ if } t \text{ is odd} \end{cases}$$

Discrete Volumes & Ehrhart Quasipolynomials

Rational polytope — convex hull of finitely many points in $\mathbb{Q}^d$

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Theorem (Ehrhart 1962) For any rational polytope $\mathcal{P} \subset \mathbb{R}^d$, $\mathrm{ehr}_{\mathcal{P}}(n) := \left| n\mathcal{P} \cap \mathbb{Z}^d \right|$ is a quasipolynomial in the integer variable n whose period divides the lcm of the denominators of the vertex coordinates of P (the denominator q of P).

Equivalently, the Ehrhart series can be written as

$$\mathrm{Ehr}_{\mathcal{P}}(z) \ := \ 1 + \sum_{n \geq 1} \mathrm{ehr}_{\mathcal{P}}(n)\, z^n \ = \ \frac{\mathrm{h}^*_{\mathcal{P}}(z)}{(1 - z^q)^{\dim \mathcal{P} + 1}}$$

Example (again) $\mathcal{P} = [-\frac{1}{2}, \frac{1}{2}]^2$ $\qquad \mathrm{Ehr}(z) = \dfrac{1 + z + 6z^2 + 6z^3 + z^4 + z^5}{(1 - z^2)^3}$

Motivation I: Really?

$\mathcal{P} \subset \mathbb{R}^d$ — rational polytope

Ehrhart functions $\operatorname{ehr}_{\mathcal{P}}(n) := \left| \mathcal{P} \cap \frac{1}{n}\mathbb{Z}^d \right| = \left| n\mathcal{P} \cap \mathbb{Z}^d \right|$ for $n \in \mathbb{Z}_{>0}$

$$\overline{\operatorname{r}}\operatorname{ehr}_{\mathcal{P}}(\lambda) := \left| \lambda\mathcal{P} \cap \mathbb{Z}^d \right| \text{ for } \lambda \in \mathbb{R}_{>0}$$

$$\operatorname{rehr}_{\mathcal{P}}(\lambda) := \left| \lambda\mathcal{P} \cap \mathbb{Z}^d \right| \text{ for } \lambda \in \mathbb{Q}_{>0}$$

Fun Fact (Linke 2011, Baldoni–Berline–Köppe–Vergne 2013, Stapledon 2017). There is an Ehrhart theory for the quasipolynomial $\overline{\operatorname{r}}\operatorname{ehr}_{\mathcal{P}}(\lambda)$ in the real variable λ.

Examples $\overline{\operatorname{r}}\operatorname{ehr}_{[1,2]}(\lambda) = \lfloor 2\lambda \rfloor - \lceil \lambda \rceil + 1 = \lambda + 1 - \{2\lambda\} - \{-\lambda\}$

$$\overline{\operatorname{r}}\operatorname{ehr}_{[-1,\frac{2}{3}]}(\lambda) = \tfrac{5}{3}\lambda + 1 - \{\tfrac{2}{3}\lambda\} - \{\lambda\}$$

Motivation II: Ehrhart Veronese

$\mathcal{P} \subset \mathbb{R}^d$ — lattice polytope

Ehrhart function $\operatorname{ehr}_{\mathcal{P}}(n) := \left| \mathcal{P} \cap \frac{1}{n}\mathbb{Z}^d \right| = \left| n\mathcal{P} \cap \mathbb{Z}^d \right|$ for $n \in \mathbb{Z}_{>0}$

Ehrhart series $\operatorname{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{n \ge 1} \operatorname{ehr}_{\mathcal{P}}(n)\, z^n = \frac{\mathrm{h}^*_{\mathcal{P}}(z)}{(1-z)^{\dim \mathcal{P}+1}}$

Fun Fact (Brenti–Welker 2009, MB–Stapledon 2010, Jochemko 2018).
The Ehrhart series of $k\mathcal{P}$ becomes nicer as k increases.

A Bit of Real Ehrhart History

Theorem (Linke 2011) Let $\mathcal{P} \subset \mathbb{R}^d$ be a rational polytope. Then $\mathrm{rehr}_{\mathcal{P}}(\lambda) = \left|\lambda\mathcal{P} \cap \mathbb{Z}^d\right|$ is a quasipolynomial in the rational variable λ whose period divides the smallest rational q such that $q\mathcal{P}$ is a lattice polytope.

Linke also proved rational analogs of the Ehrhart–Macdonald reciprocity theorem and McMullen's theorem about the periods of the coefficient functions when writing

$$\mathrm{rehr}_{\mathcal{P}}(\lambda) \ = \ c_d(\lambda)\,\lambda^d + c_{d-1}(\lambda)\,\lambda^{d-1} + \cdots + c_0(\lambda)$$

She views these coefficient functions as piecewise polynomials and proved a differential equation for them.

Baldoni–Berline–Köppe–Vergne (2013): algorithmic theory of intermediate sums on polyhedra, with $\mathrm{rehr}_{\mathcal{P}}(\lambda)$ as a special case.

A Bit of Real Ehrhart History

Motivated by motivic integration, Stapledon (2008) introduced the weighted h^*-polynomial of a lattice polytope $\mathcal{P} \subset \mathbb{R}^d$, which he later (2017) realizes via

$$1 + \sum_{\lambda \in \mathbb{Q}_{>0}} \left| \partial_{\neq 0}(\lambda \mathcal{P}) \cap \mathbb{Z}^d \right| t^\lambda \ = \ \frac{\widetilde{\mathrm{h}}_{\mathcal{P}}(t)}{(1-t)^{\dim \mathcal{P}}}$$

and uses them to compute Ehrhart polynomials of free sums, generalizing work by Braun (2006) and MB–Jayawant–McAllister (2013).

He realizes that $\widetilde{\mathrm{h}}_{\mathcal{P}}(t)$ is a polynomial in certain fractional powers of t with nonnegative coefficients. In the case that $\mathbf{0} \in \mathcal{P}^\circ$ he proves that $\widetilde{\mathrm{h}}_{\mathcal{P}}(t)$ is symmetric.

The Setup

$\mathcal{P} \subset \mathbb{R}^d$ — rational polytope $\longrightarrow$ $\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \le \mathbf{b} \right\}$

The codemoninator of $\mathcal{P}$ is $r := \operatorname{lcm}(\mathbf{b})$

Lemma (1) $\operatorname{rehr}_{\mathcal{P}}(\lambda) = \left| \lambda\mathcal{P} \cap \mathbb{Z}^d \right|$ is constant for $\lambda \in (\frac{n}{r}, \frac{n+1}{r}),\ n \in \mathbb{Z}_{\ge 0}$

(2) If $\mathbf{0} \in \mathcal{P}$ then $\operatorname{rehr}_{\mathcal{P}}(\lambda)$ is monotone

Examples $\overline{\operatorname{rehr}}_{[1,2]}(\lambda) = \lambda + 1 - \{2\lambda\} - \{-\lambda\}$

$$\overline{\operatorname{rehr}}_{[-1,\frac{2}{3}]}(\lambda) = \tfrac{5}{3}\lambda + 1 - \{\tfrac{2}{3}\lambda\} - \{\lambda\}$$

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Corollary

$$\bar{\operatorname{r}}\operatorname{ehr}_{\mathcal{P}}(\lambda) = \begin{cases} \operatorname{rehr}_{\mathcal{P}}(\lambda) & \text{if } \lambda \in \frac{1}{r}\mathbb{Z}_{\ge 0} \\ \operatorname{rehr}_{\mathcal{P}}(\lfloor \lambda \rceil) & \text{if } \lambda \notin \frac{1}{r}\mathbb{Z}_{\ge 0} \end{cases}$$

where $\lfloor \lambda \rceil := \frac{2j+1}{2r}$ for $\left| \lambda - \frac{2j+1}{2r} \right| < \frac{1}{2r}$

If $\mathbf{0} \in \mathcal{P}$ then $\bar{\operatorname{r}}\operatorname{ehr}_{\mathcal{P}}(\lambda) = \operatorname{rehr}_{\mathcal{P}}\left(\frac{\lfloor r\lambda \rfloor}{r} \right)$

The Setup

$\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \le \mathbf{b} \right\}$ with codemoninator $r := \operatorname{lcm}(\mathbf{b})$

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If $\mathbf{0} \notin \mathcal{P}$ then

$$\overline{\operatorname{rehr}}_{\mathcal{P}}(\lambda) \;=\; \begin{cases} \operatorname{rehr}_{\mathcal{P}}(\lambda) & \text{if } \lambda \in \frac{1}{r}\mathbb{Z}_{\ge 0} \\ \operatorname{rehr}_{\mathcal{P}}(\lfloor \lambda \rceil) & \text{if } \lambda \notin \frac{1}{r}\mathbb{Z}_{\ge 0} \end{cases}$$

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The Upshot $\overline{\operatorname{rehr}}_{\mathcal{P}}(\lambda)$ is determined by $\left\{ \operatorname{rehr}_{\mathcal{P}}(\lambda) : \; \lambda \in \frac{1}{2r}\mathbb{Z}_{\ge 0} \right\}$

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Rational Ehrhart Series

$\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \le \mathbf{b} \right\}$ with codemoninator $r := \operatorname{lcm}(\mathbf{b})$

(Refined) rational Ehrhart series

$$\operatorname{REhr}_{\mathcal{P}}(t) \quad := \quad 1 + \sum_{\lambda \in \frac{1}{r}\mathbb{Z}_{>0}} \operatorname{rehr}_{\mathcal{P}}(\lambda)\, t^{\lambda}$$

$$\operatorname{RREhr}_{\mathcal{P}}(t) \quad := \quad 1 + \sum_{\lambda \in \frac{1}{2r}\mathbb{Z}_{>0}} \operatorname{rehr}_{\mathcal{P}}(\lambda)\, t^{\lambda}$$

Examples

$$\operatorname{REhr}_{[-1,\frac{2}{3}]}(t) \quad = \quad \frac{1 + t^{\frac{1}{2}} + t + t^{\frac{3}{2}} + t^2}{(1-t)(1-t^{\frac{3}{2}})}$$

$$\operatorname{RREhr}_{[1,2]}(t) \quad = \quad \frac{1 + t^{\frac{1}{2}} + t^{\frac{3}{4}} + t^{\frac{5}{4}}}{(1-t)^2}$$

Rational Ehrhart Series

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(Refined) rational Ehrhart series

$$\begin{aligned} \operatorname{REhr}_{\mathcal{P}}(t) &:= 1 + \sum_{\lambda \in \frac{1}{r}\mathbb{Z}_{>0}} \operatorname{rehr}_{\mathcal{P}}(\lambda)\, t^{\lambda} \\ \operatorname{RREhr}_{\mathcal{P}}(t) &:= 1 + \sum_{\lambda \in \frac{1}{2r}\mathbb{Z}_{>0}} \operatorname{rehr}_{\mathcal{P}}(\lambda)\, t^{\lambda} \end{aligned}$$

Theorem Let $m \in \mathbb{Z}_{>0}$ such that $\frac{m}{r}\mathcal{P}$ is a lattice polytope. Then

$$\operatorname{REhr}_{\mathcal{P}}(t) = \frac{\operatorname{rh}^*_{\mathcal{P}}(t)}{\left(1 - t^{\frac{m}{r}}\right)^{d+1}}$$

where $\operatorname{rh}^*_{\mathcal{P}}(t)$ is a polynomial in $\mathbb{Z}\left[t^{\frac{1}{r}}\right]$ with nonnegative coefficients.

Rational Ehrhart Series

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where $\operatorname{rh}^*_{\mathcal{P}}(t)$ is a polynomial in $\mathbb{Z}\left[t^{\frac{1}{r}}\right]$ with nonnegative coefficients.

Corollary The period of the quasipolynomial $\operatorname{rehr}_{\mathcal{P}}(\lambda)$ divides $\frac{m}{r}$.

Similar results hold for $\operatorname{RREhr}_{\mathcal{P}}(t)$, with r replaced by $2r$.

Corollary (Linke 2011) Let $\mathcal{P}$ be a lattice polytope. Then $\overline{\operatorname{r}}\operatorname{ehr}_{\mathcal{P}}(\lambda)$ and $\operatorname{rehr}_{\mathcal{P}}(\lambda)$ are quasipolynomials of period 1.

Rational Ehrhart Series

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where $\mathrm{rh}^*_{\mathcal{P}}(t)$ is a polynomial in $\mathbb{Z}\left[t^{\frac{1}{r}}\right]$ with nonnegative coefficients.

Corollary (Linke 2011) For a rational polytope $\mathcal{P}$, $(-1)^{\dim \mathcal{P}}\, \mathrm{\overline{r}ehr}_{\mathcal{P}}(-\lambda)$ equals the number of interior lattice points in $\lambda\mathcal{P}$, for any $\lambda > 0$.

Remark If $\frac{m}{r} \in \mathbb{Z}$ then $\mathrm{h}^*_{\mathcal{P}}(t) = \mathrm{Int}\left[\mathrm{rh}^*_{\mathcal{P}}(t)\right]$.

Gorenstein Musings

Theorem (Hibi 1991, Fiset–Kasprzyk 2008) Let $\mathcal{P}$ be a rational polytope with $\mathbf{0} \in \mathcal{P}^\circ$. If the polar dual of $\mathcal{P}$ is a lattice polytope then $\mathrm{h}^*_{\mathcal{P}}(t)$ is symmetric.

(This fits, more generally, with Stanley's theory of Gorenstein rings.)

Theorem Let $\mathcal{P}$ be a rational polytope with $\mathbf{0} \in \mathcal{P}^\circ$. Then $\mathrm{rh}^*_{\mathcal{P}}(t)$ is symmetric.

Corollary Let $\mathcal{P}$ be a rational polytope with $\mathbf{0} \in \mathcal{P}^\circ$. Then $\mathrm{h}^*_{\mathcal{P}}(t)$ is the Veronese of a symmetric polynomial.

Example $\mathrm{REhr}_{[-1,\frac{2}{3}]}(t) =$

$$\frac{1 + t^{\frac{1}{2}} + 2t + 3t^{\frac{3}{2}} + 4t^2 + 4t^{\frac{5}{2}} + 4t^3 + 4t^{\frac{7}{2}} + 3t^4 + 2t^{\frac{9}{2}} + t^5 + t^{\frac{11}{2}}}{(1-t^3)^2}$$

Symmetric Decompositions

Theorem Let $\mathcal{P}$ be a rational polytope with $\mathbf{0} \in \mathcal{P}^\circ$. Then $\mathrm{rh}^*_{\mathcal{P}}(t)$ is symmetric.

Corollary (Betke–McMullen 1985, MB–Braun–Vindas-Meléndez 2021+) Let $\mathcal{P}$ be a rational polytope with denominator k and $\mathbf{0} \in \mathcal{P}^\circ$. Then there exist polynomials $a(t)$ and $b(t)$ with nonnegative coefficients such that

$$\mathrm{h}^*_{\mathcal{P}}(t) = a(t) + t\,b(t)\,, \quad t^{k(d+1)-1}\,a(\tfrac{1}{t}) = a(t)\,, \quad t^{k(d+1)-2}\,b(\tfrac{1}{t}) = b(t)\,.$$

Proof Idea ($k=1$):

$$\mathrm{REhr}_{\mathcal{P}}(t) \;=\; \frac{\mathrm{h}^*_{\partial(\frac{1}{r}\mathcal{P})}\left(t^{\frac{1}{r}}\right)}{\left(1-t^{\frac{1}{r}}\right)(1-t)^d}$$

Note that $\mathrm{h}^*_{\partial(\frac{1}{r}\mathcal{P})}(t)$ is symmetric and nonnegative, and

$$\begin{aligned}\mathrm{rh}^*_{\mathcal{P}}(t) \;&=\; \mathrm{Int}\left[\left(1+t^{\frac{1}{r}}+\cdots+t^{\frac{r-1}{r}}\right)\mathrm{h}^*_{\partial(\frac{1}{r}\mathcal{P})}\left(t^{\frac{1}{r}}\right)\right] \\ &=\; \mathrm{Int}\left[\mathrm{h}^*_{\partial(\frac{1}{r}\mathcal{P})}\left(t^{\frac{1}{r}}\right)\right] + \mathrm{Int}\left[\left(t^{\frac{1}{r}}+t^{\frac{2}{r}}+\cdots+t^{\frac{r-1}{r}}\right)\mathrm{h}^*_{\partial(\frac{1}{r}\mathcal{P})}\left(t^{\frac{1}{r}}\right)\right]\end{aligned}$$

A Remark on Complexity

$\mathcal{P} \subset \mathbb{R}^d$ — rational polytope with codemoninator r

To capture $\operatorname{rehr}_{\mathcal{P}}(\lambda)$ (or $\overline{\operatorname{rehr}}_{\mathcal{P}}(\lambda)$), we need to compute...

$$\mathbf{0} \notin \mathcal{P} \quad \longrightarrow \quad \operatorname{RREhr}_{\mathcal{P}}(t) \; = \; 1 + \sum_{\lambda \in \frac{1}{2r}\mathbb{Z}_{>0}} \operatorname{rehr}_{\mathcal{P}}(\lambda)\, t^{\lambda} \; = \; \frac{\operatorname{rrh}^*_{\mathcal{P}}(t)}{(1-t^q)^{d+1}}$$

$$\mathbf{0} \in \partial\mathcal{P} \quad \longrightarrow \quad \operatorname{REhr}_{\mathcal{P}}(t) \; = \; 1 + \sum_{\lambda \in \frac{1}{r}\mathbb{Z}_{>0}} \operatorname{rehr}_{\mathcal{P}}(\lambda)\, t^{\lambda} \; = \; \frac{\operatorname{rh}^*_{\mathcal{P}}(t)}{(1-t^q)^{d+1}}$$

$\mathbf{0} \in \mathcal{P}^{\circ} \quad \longrightarrow \quad \operatorname{rh}^*_{\mathcal{P}}(t)$ symmetric