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The size of the minimal automaton for an algebraic sequence

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Let $s(n)_{n\geq 0}$ be a sequence whose terms are elements of a finite field $\mathbb{F}_q$. A major theorem of Christol [2, 3] states that $s(n)_{n\geq 0}$ is algebraic if and only if it is q-automatic. That is, there exists a nonzero polynomial $P(x, y) \in \mathbb{F}_q[x, y]$ such that $P(x, \sum_{n\geq 0} s(n)x^n) = 0$ if and only if there is a finite automaton that outputs $s(n)$ when fed the base-q digits of n (say, starting with the least significant digit).

We therefore have two quite different ways of representing q-automatic sequences — polynomials and automata. A natural question is how the size of the minimal polynomial for a sequence (measured by its x-degree and y-degree) relates to the size of the minimal automaton for the sequence (measured by the number of states), and vice versa.

Given an algebraic series $\sum_{n\geq 0} s(n)x^n$ specified by a polynomial $P(x, y)$ with x-degree h, y-degree d, and genus g, Bridy [1] used algebraic geometry techniques to obtain the upper bound $(1 + o(1))q^{h+d+g-1}$ on the number of states in the minimal automaton generating $s(n)_{n\geq 0}$, where $o(1)$ tends to 0 as any of q, h, d, g gets large.

We show that progress can be made toward this bound without using tools from algebraic geometry, by analyzing orbits of certain linear operators on a finite-dimensional vector space of bivariate polynomials.

Keywords
automatic sequence, algebraic sequence, Christol's theorem

References

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