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Constructing minimal telescopers for rational functions in three discrete variables *

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Creative telescoping is a powerful method pioneered by Zeilberger [5, 6] in the 1990s and has now become the cornerstone for finding closed forms for definite sums (and definite integrals) in computer algebra. In the case of summation, specialized to the trivariate case, in order to compute a sum of the form $\sum_{y=a_1}^{b_1}\sum_{z=a_2}^{b_2} f(x,y,z)$, the main task of creative telescoping consists in finding rational functions (or polynomials) $c_0,\ldots,c_\rho$ in x, not all zero, and two functions $g(x,y,z), h(x,y,z)$ in the same domain as f such that

$$\begin{aligned} &c_0(x)f(x,y,z)+c_1(x)f(x+1,y,z)+\cdots+c_\rho(x)f(x+\rho,y,z)\\ &=g(x,y+1,z)-g(x,y,z)+h(x,y,z+1)-h(x,y,z). \end{aligned} \tag{1}$$

We call the nonzero operator $L=c_\rho S_x^\rho+\cdots+c_1S_x+c_0$ with S_x being the shift operator in x a *telescoper* for f and the pair (g,h) a *certificate* for L.

Various algorithmic generalizations and improvements for the method of creative telescoping have been developed over the past two decades. At the present time, the reduction-based approach has gained the most support as it is both efficient in practice and has the important feature of being flexible to find a telescoper for a given special function with or without construction of a certificate. This is desirable in the typical situation where only the telescoper is of interest and its size is much smaller than that of the certificate.

In this talk, we describe a recent algorithm developed by the authors in [2] for constructing minimal telescopers for rational functions in three discrete variables. This is the first step toward developing reduction-based creative telescoping algorithms for special functions having more than two discrete variables.

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As with other reduction-based algorithms, our starting point is to find a suitable reduction for trivariate rational functions. In particular, based on Hou and Wang's work in [4], along with the use of difference homomorphisms, we extend the Abramov reduction [1] for determining summability of univariate rational functions to bivariate ones. This extended reduction brings the given rational function $f(x, y, z)$ to another rational function $\mathrm{red}(f)$ modulo summable ones, namely rational functions admitting the form as the right-hand side of (1), where $\mathrm{red}(f)$ satisfies: (i) $\mathrm{red}(f) = 0$ whenever f is summable, and (ii) $\mathrm{red}(f)$ is minimal in certain sense. Such a $\mathrm{red}(f)$ is unique up to congruence modulo summable rational functions. In order to find a telescoper for f, we then iteratively compute $\mathrm{red}(f), \mathrm{red}(S_x(f)), \mathrm{red}(S_x^2(f)), \ldots$ until we find rational functions $c_0, \ldots, c_\rho$ in x, not all zero, such that

$$c_0 \,\mathrm{red}(f) + \cdots + c_\rho \,\mathrm{red}(S_x^\rho(f)) \equiv 0 \mod (\text{summable rational functions}).$$

By showing that the expression on the left-hand side is congruent to $\mathrm{red}(c_0 f + \cdots + c_\rho S_x^\rho(f))$, we conclude from the minimality of $\mathrm{red}(\cdot)$ that $\mathrm{red}(c_0 f + \cdots + c_\rho S_x^\rho(f)) = 0$. This thus implies that $c_0 f + \cdots + c_\rho S_x^\rho(f)$ is summable, yielding a telescoper $c_0 + \cdots + c_\rho S_x^\rho$ for f. Note that the first nontrivial linear dependency leads to a telelscoper of minimal order, and the termination of our algorithm is guaranteed by a known existence criterion of telescopers developed in [3].

One can tell from the above description that our algorithm finds a minimal telescoper for a given trivariate rational function without also needing to compute an associated certificate. Computational experiments will be provided so as to illustrate the efficiency of our algorithm.

Keywords
Abramov reduction, Telescoper, Difference homomorphism

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