Apparent Singularities of D-finite Systems

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A D-finite function is specified by a linear ordinary differential equation with polynomial coefficients and finitely many initial values. Every singularity of a D-finite function will be a root of the coefficient of the highest order derivative appearing in the corresponding differential equation. For instance, x^{-1} is a solution of the equation $xf'(x)+f(x)=0$, and the singularity at the origin is also the root of the polynomial x. However, the converse is not true. For example, the solution space of the differential equation $xf'(x)-4f(x)=0$ is spanned by x^4 as a vector space, but none of those functions has singularity at the origin.

More specifically, for an ordinary equation $p_0(x)f(x)+\cdots+p_r(x)f^{(r)}(x)=0$ with polynomial coefficients $p_1,\ldots,p_r$ and $p_r\neq 0$, the roots of p_r are called the singularities of the equation. A root α of p_r is call *apparent* if the differential equation admits r linearly independent formal power series solutions in $x-\alpha$. Deciding whether a singularity is apparent is therefore the same as checking whether the equation admits a fundamental system of formal power series solutions at this point. This can be done by inspecting the so-called *indicial polynomial* of the equation at α and solving a system of finitely many linear equations. If a singularity α of an ordinary differential is apparent, then we can always construct a second ordinary differential equation whose solution space contains all the solutions of the first equation, and which does not have α as a singularity any more. This process is called *desingularization*.

The purpose of our work is to generalize the facts sketched above to the multivariate setting. Instead of an ODE, we consider systems of PDEs known as D-finite systems. A D-finite system is a finite set of linear homogeneous partial differential equations with polynomial coefficients in several variables, whose solution space is of finite dimension. For such systems, we define the notion of a singularity in terms of the polynomials appearing in them. We show that a point is a singularity of the system unless it admits a basis of power series solutions in which the starting monomials are as small as possible with respect to some term order. Then a singularity is apparent if the system admits a full basis of power series solutions, the starting terms of which are not as small as possible. We then prove that apparent singularities can be removed like in the univariate case by adding suitable additional solutions to the system at hand. The details can be found in [1].

References

[1] Y. Zhang, *Univariate contraction and multivariate desingularization of Ore ideals*, PhD thesis, Institute for Algebra, Johannes Kepler Univ., (2017).
http://www.algebra.uni-linz.ac.at/people/yzhang/yzhang_PhDthesis_final.pdf