Computer Algebra Algorithms for Proving Jacobi Theta Function Identities

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Many number theorists, e.g., Ramanujan, Hardy, Rademacher, Berndt, Borwein, etc., have proved a substantial amount of theta function relations by hand (see [1]–[8]). There was no general method for proving such relations, and the computation in their proofs are usually tedious.

Example 1. *[7, (93.22)]*

$$\theta_3^{(4)}(0|\tau)\theta_3(0|\tau)-3(\theta_3''(0|\tau))^2-2\theta_3(0|\tau)^2\theta_2(0|\tau)^4\theta_4(0|\tau)^4 \equiv {}^1 0.$$

Example 2. *[5, p. 17]*

$$\sum_{j=1}^{4}\theta_j(x|\tau)\theta_j(y|\tau)\theta_j(u|\tau)\theta_j(v|\tau)-2\theta_3(x_1|\tau)\theta_3(y_1|\tau)\theta_3(u_1|\tau)\theta_3(v_1|\tau)\equiv 0,$$

where $x_1 := \frac{1}{2}(x+y+u+v)$ *and* $y_1 := \frac{1}{2}(x+y-u-v)$, $u_1 := \frac{1}{2}(x-y+u-v)$ *and* $v_1 := \frac{1}{2}(x-y-u+v)$.

Example 3. *[3, p. 218] A form of the cubic modular equation is*

$$\theta_3(0|\tau)\theta_3(0|3\tau)-\theta_4(0|\tau)\theta_4(0|3\tau)-\theta_2(0|\tau)\theta_2(0|3\tau)\equiv 0.$$

Example 4. *[1, p. 285] Let* $\eta(\tau) := e^{\pi i\tau/12}\prod_{k=1}^{\infty}(1-e^{2\pi i\tau k})$. *Then*

$$\theta_3(0|\tau)^2\theta_3(0|5\tau)^2-\theta_2(0|\tau)^2\theta_2(0|5\tau)^2-\theta_4(0|\tau)^2\theta_4(0|5\tau)^2\equiv 8\eta(2\tau)^2\eta(10\tau)^2.$$

By using such theta function relations, several important results can be obtained. For instance, by using Example 1, Rademacher derived the formula for the number of presentations of a natural number as a sum of 10 squares. Moreover, those types of relations also play an important role in physics and in the evaluation of π.

[1] We use the notation $f_1(z_1,z_2,\dots)\equiv f_2(z_1,z_2,\dots)$ if we want to emphasize that the equality between the functions holds for all possible choices of the arguments z_j.

Our goal is to automatize the proving procedures of relations and the discovery of relations. As a first step, in [9] we provided an algorithm to prove identities involving

$$\theta_j^{(k)}(0|\tau) := \frac{\partial^k \theta_j}{\partial z^k}(z|\tau)\Big|_{z=0}, \quad k \in \mathbb{N} := \{0,1,2,\dots\}.$$

Then, in [10], we extend the function space in [9] and provided two algorithms to prove identities in the form of

$$\sum c(i_1,i_2,i_3,i_4)\theta_1(z|\tau)^{i_1}\theta_2(z|\tau)^{i_2}\theta_3(z|\tau)^{i_3}\theta_4(z|\tau)^{i_4} \equiv 0$$

with $c(i_1,i_2,i_3,i_4) \in \mathbb{K}[\Theta]$, where $\mathbb{K}$ is a computable field and

$$\Theta := \left\{\theta_1^{(2k+1)}(0|\tau) : k \in \mathbb{N}\right\} \cup \left\{\theta_j^{(2k)}(0|\tau) : k \in \mathbb{N} \text{ and } j = 2,3,4\right\}.$$

In addition, by our approach, we can also produce two general classes of relations. In this talk we will briefly show the essence of our methods for [9] and [10], which is mainly based on modular form techniques and the theory of elliptic functions. We will also demonstrate our Mathematica package "ThetaFunctions".

References

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