Time for the New Ansatz (?)

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Mathematics is a science of describing patterns. It is a commonly known technique to describe the patterns of sequences using recurrence relations, both by using constant coefficients (aka C-finite ansatz, [1, 4]) i.e. the sequence $\{a(n)\}_{n=0}^{\infty}$ where there are constants $c_0, c_1, \dots, c_{k-2}, c_{k-1}$ such that

$$c_0 a(n) + c_1 a(n+1) + \cdots + c_{k-1} a(n+k-1) + a(n+k) = 0, \quad \text{for all } n \geq 0,$$

or by using polynomial coefficients (aka holonomic ansatz, [1, 3]) i.e. the sequence $\{b(n)\}_{n=0}^{\infty}$ where there are polynomials $p_0(n), p_1(n), p_2(n), \dots, p_{k-1}(n), p_k(n)$ with $p_k(n) \neq 0$, such that

$$p_0(n)b(n) + p_1(n)b(n+1) + \cdots + p_k(n)b(n+k) = 0, \quad \text{for all } n \geq 0.$$

However there are still many important sequences that do not belong to these classes. The first example is the (Somos) sequence defined by a complicated looking non-linear recurrence relation:

$$a(n)(a(n+1)\cdot a(n+3) - a(n+2)^2) - a(n+2)\cdot a(n+1)^2 = 0, \quad \text{for all } n \geq 0$$

where $a(0) = 1, a(1) = 1$ and $a(2) = 2$.

Here are the first ten terms of the sequence:

$$1, 1, 2, 6, 30, 240, 3120, 65520, 2227680, 122522400$$

This sequence is growing too fast to be C-finite or holonomic, but still simple enough for a human to detect the pattern. This strongly suggests us to create a new ansatz for this type of sequences.

The second example came up when I worked on Schmidt's number, [2]. This is part of the main theorem. For $k \geq 0$ and $r \geq 1$, define $a_{k,j}^{(r)}$ as follows:

$$\binom{n}{k}^r \binom{n+k}{k}^r = \sum_j a_{k,j}^{(r)} \binom{n}{j}\binom{n+j}{j}.$$

It is not clear at all that this multi-dimensional sequence $a_{k,j}^{(r)}$ are integers until we discover the non-holonomic recurrence relation of $a_{k,j}^{(r)}$:

$a_{k,k}^{(1)} = 1$, $a_{k,j}^{(1)} = 0$ ($j \neq k$) and

$$a_{k,j}^{(r+1)} = \sum_i \binom{k+i}{i} \binom{k}{j-i} \binom{j}{k} a_{k,i}^{(r)}.$$

In conclusion, we will explore many of these examples and propose some new types of ansatz accordingly.

References

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