External Littelmann Paths

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Applications of Computer Algebra-Algorithmic Combinatorics
Jerusalem College of Technology
July 2017

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The Problem

- The irreducible modules for the symetric groups over $\mathbb{C}$ are labelled by partitions.
- Over a field of characteristic p,the irreducible modules are labelled by p-regular partitions.
- For cyclotomic Hecke algebras, the irreducible modules are labelled by e-regular multipartitions.

The problem: We have only a recursive algorithm for constructing e-regular multipartitions.

Affine Lie Algebras of Type A

- $\mathcal{G}$ - an affine Lie algebra of Type A,
- Dynkin diagram a circle,
- $\Lambda_0, \Lambda_1, \ldots, \Lambda_{e-1}$ - fundamental weights,
- $\alpha_0, \alpha_1, \ldots, \alpha_{e-1}$-simple roots,
- $\delta = \sum \alpha_i$ - the null root.
- $\mathbb{Q}_+ = \{\alpha = \sum c_i\alpha_i\}$, with content $(c_0, c_1, \ldots, c_{e-1})$,
- $def(\Lambda - \alpha) = (\Lambda \mid \alpha) - \frac{1}{2}(\alpha \mid \alpha)$,
- The corank 1 Cartan matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & \ldots & 0 & -1 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & -1 & 2 & -1 \\ -1 & 0 & \ldots & 0 & -1 & 2 \end{bmatrix}$$

Kashiwara crystals

- Let $e_i, f_i, h_i, i = 0, 1, \ldots, e-1$ be a Chevalley basis
- Let $\Lambda = a_0\Lambda_0 + \ldots a_{e-1}\Lambda_{e-1}, a_i \in \mathbb{Z}_+$
- Let $V(\Lambda)$ be a highest weight representation generated by the f_i from $u_\emptyset$ of weight Λ
- Let $P(\Lambda)$ be the sets of weights of weight spaces of $V(\Lambda)$
- A Kashiwara *crystal* $B(\Lambda)$ is a labeling of the basis of $V(\Lambda)$ with operations e_i and f_i

Kashiwara crystals of Type A

In Type A, the level r of Λ is $a_0 + \cdots + a_{e-1}$. There are three important versions of the Kashwara crystal in type A:

- by e-regular multipartions, sets of r partitions with no e rows repeated,
- by Littelmann paths, which we will soon describe in greater detail,
- and by canonical basis elements, which are q-polynomials in a space called Fock space, coming from physics.

Kashiwara crystals of Type A

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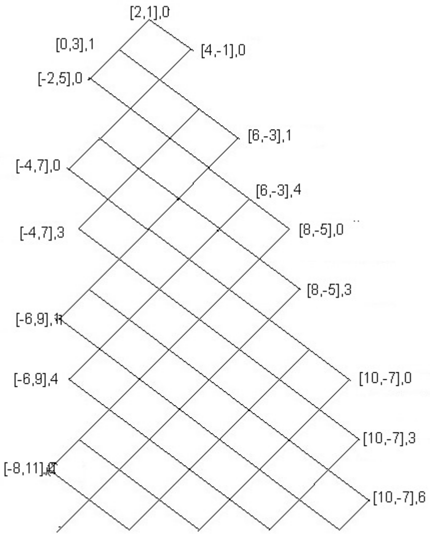
In general all three are generated recursively, using computer algebra programs. Our general research program concerns the combinatorial relations among all three, but for this talk, we focus on the possibility of passing directly between the e-regular multipartitions and the Littelmann paths.

The reduced crystal

We get the *reduced* crystal with vertices $P(\Lambda)$ by adding edges whereever there is an edge in the underlying Kashwara crystal, where we take all i-strings parallel to each other. The weights in $P(\Lambda)$ are of the form $\lambda = \Lambda - \alpha$ for some α. The highest-weight representation being integrable, all i-strings are of finite length. To each vertex of $P(\Lambda)$ we associate

- The content of α
- The defect
- The hub θ, where $\theta_i =< h_i, \lambda >$

Reduced crystal, $e = 2, \Lambda = 2\Lambda_0 + \Lambda_1$, with hubs



Categorification in Type A

Chuang and Rouquier proved in [CR] that the highest weight representation $V(\Lambda)$ has a categorification to a direct sum of *cyclotomic Hecke algebras* H_n^{Λ}, where the degree n runs from 0 out toward ∞. The Chevalley generators e_i, f_i are categorified to restriction and induction functors E_i, F_i, the weight spaces correspond to blocks, simple relections from the Weyl group correspond to derived equivalences, and the elements in the Kashwara crystal correspond to simple modules. The simplest but best known example is for $r = 1$, $\Lambda = \Lambda_0$, over a field of characteristic e, where the simple modules of the symmetric groups correspond to e-regular partitions.

The Littelmann path model for $B(\Lambda)$

An LS-path $\pi(t)$ is a piecewise linear path in the weight space of the Lie algebra $\mathcal{G}$,

$$\langle =< \Lambda_0, \Lambda_1 \ldots, \Lambda_{e-1}, \delta >$$

and parameterized by the real interval $[0, 1]$, with $\pi(0) = 0$. Littelmann [L] proved that the set of paths obtained by acting with various f_i , starting with the path from 0 to Λ, is in one-to-one correspondence with the basis elements of the highest weight representation $V(\Lambda)$, and has the structure of a Kashiwara crystal. The straight segments in the piecewise linear paths are rational multiples of weight vectors in the orbit of Λ under the action of the Weyl group W, which we will call defect zero weight vectors. The corner points are the endpoints of these straight segments, the final corner point $\pi(1) \in P(\Lambda)$ being the weight of the basis element in $V(\Lambda)$.

The Littelmann path model for $B(\Lambda)$

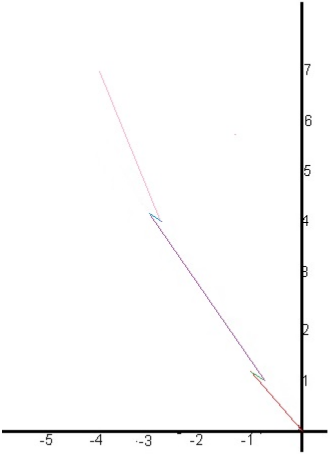
Littelmann proved in [L] that a Littelman path corresponding to a crystal base element is integral, in the sense that for each i the lowest i-coordinate among all the corner points is an integer.

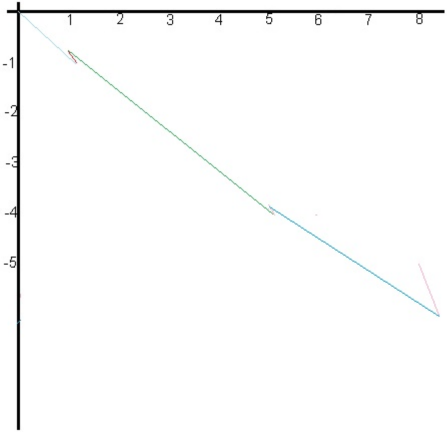
Definition

A Littelmann path $\pi(t)$ has an *LS*-representation if there is a sequence of defect 0 weights $\nu_p, \ldots, \nu_0$ and rational number $a_{p+1} = 0, a_p, \ldots, a_0 = 1$ such that for $t \in [a_{i+1}, a_i]$, we have

$$\pi(t) = \pi(a_{i+1}) + (t - a_{i+1})\nu_i$$

Example





$[(5,2,1),(1),\emptyset]; [(6,3,2,1),(2),\emptyset]$

Multipartitions

In order to generate the e-regular multipartiitions, we must choose an ordering of the fundamental weights in Λ,

$$\Lambda = \Lambda_{k_1} + \cdots + \Lambda_{k_r}$$

We will follow Mathas in [M] in requiring $k_1 \leq k_2 \leq \cdots \leq k_r$. We can then summarize by setting

$$\Lambda = a_0\Lambda_0 + \cdots + a_{e-1}\Lambda_{e-1}$$

The Young diagram of a defect 0 weight λ will be represented by $Y(\lambda)$. If the ℓ-th subpartition of λ is nonempty, then we associate to each node in the Young diagram a residue, where the node (i, j) is given residue

$$k_\ell + j - i$$

This will be called a k_ℓ-corner subpartition.

Multipartitions

There is a recursive algorithm for generating e-regular multipartitions.

0	1	2	0	1	2	0
2	0	1				

We write the signature of addable and removable nodes for a given residue. For 1 we would get "+-+". After removing all "-+", we take the leftmost "+'. and add it, getting

0	1	2	0	1	2	0
2	0	1				
1						

Standard Littelmann paths

A Littelmann path will be called *standard* if the rational numbers are of the form

$$e_m = \frac{c_m}{d_m}, \tag{1}$$

where d_m was the number of nodes added to a defect 0 multipartion with first row $m - 1$ to get that for m. Similarly c_m is the number of nodes added of that residue among those making up the d_m in the defect 0 multipartition.

Computer resouces for crystals

The Littelmann paths for a given $\mathcal{G}$ and Λ can be generated in Sagemath using the function CrystalOfLSPaths() written by Mark Shimozono and Anne Schilling. In addition, Travis Scrimshaw recently implemented an algorithm of Matt Fayers to calculate the canonical basis, named FockSpace().
Our own modification computes the following for basis element b:

- The multipartition
- The Littelmann path and optionally, the corner-points
- The canonical baisis element
- The set of paths in the reduced crystal leading to b

Residue-homogeneous multipartitions

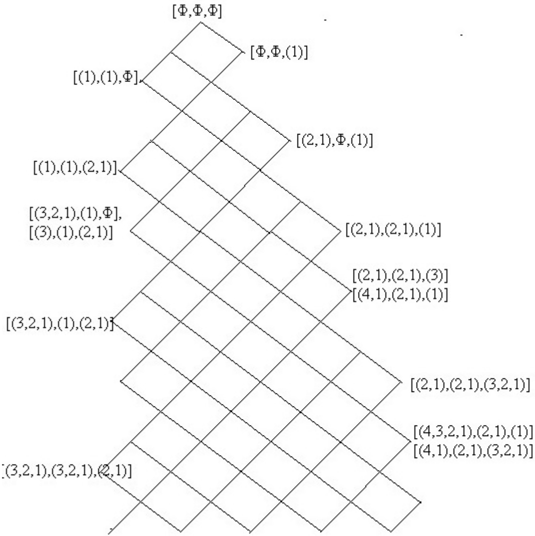
The following codiiton will ensure that the end points of all the rows would have the same residue 0 or 1.

Definition

A multipartition will be called *residue homogeneous* if it satisfies the following conditions:

- each partition has rows of alternating parity,
- all zero corner partitions have first rows of the same parity and the 1-corner of opposite parity,
- all non-zero partitions except possible the last end with a singleton
- The length of the first row is less than or equal to the length of the previous column.

Reduced crystal, $e = 2, \Lambda = 2\Lambda_0 + \Lambda_1$, with multipartitions



The Main Result

By a result of Mathas, [M], the first condition, for $e = 2$, ensures that the multipartition is e-regular.

Theorem

In the case $e = 2$, the Littelmann path corresponding to a residue homogeneous multipartition is standard

The set of all residue homogeneous multipartitons can be determined non-recursively, and then the corresponding Littelmann path constructed, which proves that the multipartitions was e-regular.

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Thank you.

Thank you.

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