External Littelmann paths for crystals of Type A

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Affine Lie algebras of type A and their highest weight representations are important in physics. They correspond to the symmetric group, the most important of the reflection groups. The basis elements of a highest weight representation with highest weight Λ of level r, organized into a Kashiwara crystal, correspond to the simple modules of the cyclotomic Hecke algebras of weight Λ and have three combinatorial representations: as multipartitions, as Littelmann paths and as canonical basis elements.

We wrote a computer program in Sage which calculated all three of these combinatorial representations simultaneously for the beginning degrees of a Kashiwara crystal. The program slows down at around degree 16, so most of our examples are in the range up to 16. We began with the case of rank $e = 2$, for which the multipartitions corresponding to basis elements, called the e-regular multipartions, are completely understood by work of Mathas [M]. We succeeded in finding a direct connection between the multipartitions at the corners of the Kashiwara crystal, which we called extremal, and Littelmann paths of a type we call standard.

Following Mathas, we write

$$\Lambda = a\Lambda_0 + b\Lambda_1. \tag{1}$$

We started with the easy case $r = 1$, and by constructing an object called the block-reduced crystal graph [AS], discovered that the corner points were alternating, i.e., had odd and even length rows alternating. Defining segment boundaries when the differences were more than one, we were able to find a representation of the external Littelmann paths which depended on the length of the first row of the segment and the distance to the top of the partitions.

A Littelman path [L] is a piecewise linear path from the unit interval to the weight space, represented in the computer by a sequence of vectors called defect 0 weights, together with coefficients which are rational numbers and determine the endpoints of the piecewise linear subpaths of the Littelmann path. The first and last vectors are called the ceiling and the floor [AKT]. We were able to show that there were no gaps between the ceiling and the floor and give exact formulae for the coefficients. For a segment i, we let b_i be the distance from the top of the partition to the bottom of the segment, and let n'_i be the number we would get if the top row of the segment is continued up in a triangular fashion to the top row. Then we get parameter boundaries $\frac{b_i}{m}$ for m with $n'_i \geq m > n'_{i+1}$. The paths had an interesting structure: long paths where the segments were being widened, and short oscillating paths where the segments were being deepened.

We then turned to the case of $r > 1$, which was considerably more challenging. However, we were helped along by the intuition we had gained from working with the $r = 1$ case. We again divided the multipartition into segments, but now a segment could contain more than one subpartition. We replaced the alternating condition with a condition we called "residue homogeneous", which ensured that the end points of all the rows would have the same residue 0 or 1. We no longer had a simple, gapless Littelmann path between ceiling and floor. To deal with this situation, we defined a multipartition which we called a pseudo-floor, which was a defect 0 partition truncated by replacing some of the subpartitions by the emply partition. We believe this object to be new.

The induction for the r >1 case started by constructing the Littelmann path for the pseudo-floor of the highest segment and began adding segments going downward. The resulting Littelmann paths, projected onto the hubs, looked very similar to the paths we had found for the $r = 1$ case, except that the end was quirky because of the pseudo-floor.

Finally, the rational numbers which gave the boundaries for the parametrization were also more complicated. Each was of the form

$$e_m = \frac{c_m}{d_m}, \tag{2}$$

where d_m was the number of nodes added to a defect 0 multipartion with first row $m-1$ to get that for m. Similarly c_m is the number of nodes added to widen the segment. Standard Littelmann paths have parameter boundaries in this form and are quite common, as we found from our experimental work on the case $e = 3$. In the general case they usually had gaps, which occurred when $e_m = e_{m+1}$. There is no known non-recursive criterion for e-regular multipartitions for $e = 3$ and level $r > 3$. We are hoping to get results in this direction for the external basis elements.

References

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