Automatic proofs for establishing the structure of integer sequences avoiding a pattern

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Is there an infinite sequence on the alphabet $\{0,1,2\}$ containing no block that occurs twice consecutively? Questions like this were investigated a century ago by the Norwegian mathematician Axel Thue, who produced some of the earliest results in combinatorics on words. If a pattern is avoidable on a given alphabet, it is natural to ask about the lexicographically least sequence that avoids the pattern. Occasionally the structure of this sequence can be discovered and proved by hand. But for many patterns this sequence is sufficiently complex that computer-assisted discovery, followed by automated proofs, seems to be necessary to make any progress.

Here we are interested in the lexicographically least integer sequence avoiding a given fractional power. Let a and b be relatively prime positive integers with $\frac{a}{b} > 1$. We say that a word w is an $\frac{a}{b}$*-power* if w can be written $v^e x$ where e is a non-negative integer, x is a prefix of v, and $|w|/|v| = a/b$. For example, $011101 = (0111)^{3/2}$ is a $\frac{3}{2}$-power. A sequence is $\frac{a}{b}$*-power-free* if none of its nonempty factors are $\frac{a}{b}$-powers. Avoiding $\frac{3}{2}$-powers, for example, means avoiding factors xyx where $|x| = |y| \geq 1$.

Notation. Let $\mathbf{s}_{a/b}$ denote the lexicographically least $\frac{a}{b}$-power-free infinite sequence on the alphabet $\mathbb{Z}_{\geq 0}$.

Guay-Paquet and Shallit [2] described the structure of the lexicographically least square-free sequence

$$\mathbf{s}_2 = 0102010301020104010201030102010 5\cdots.$$

More generally, for an integer $a \geq 2$ we have $\mathbf{s}_a = \varphi^\infty(0)$, where $\varphi : \mathbb{Z}_{\geq 0}^* \to \mathbb{Z}_{\geq 0}^*$ is the morphism defined by $\varphi(n) = 0^{a-1}(n+1)$. Rowland and Shallit [4] gave a recurrence for

$$\mathbf{s}_{3/2} = 001102\,100112\,001103\,100113\,001102\,100114\,001103\,100112\cdots.$$

The sequence $\mathbf{s}_{3/2}$ is 6-regular in the sense of Allouche and Shallit [1]; informally, this means that the ith term can be computed directly from the base-6 digits of i.

Significant motivation for the present study is to put this '6' into context by studying $\mathbf{s}_{a/b}$ systematically. We show that for many rational numbers $\frac{a}{b}$, the sequence $\mathbf{s}_{a/b}$ is the fixed point of a k-uniform morphism for some integer k. (A morphism φ on an alphabet Σ is *k-uniform* if $|\varphi(n)| = k$ for all $n \in \Sigma$.)

For example, consider

$$\mathbf{s}_{5/3} = 0000101\,0000101\,0000101\,0000101\,0000102\,0000101\,0000102\cdots.$$

This sequence belongs to an infinite family of sequences, all generated by similar morphisms.

Theorem. *Let a, b be relatively prime positive integers such that $\frac{5}{3} \leq \frac{a}{b} < 2$ and $\gcd(b,2) = 1$. Let φ be the $(2a-b)$-uniform morphism defined by*

$$\varphi(n) = 0^{a-1}\,1\,0^{a-b-1}\,(n+1)$$

for all $n \in \mathbb{Z}_{\geq 0}$. Then $\mathbf{s}_{a/b} = \varphi^{\infty}(0)$.

There are two steps in the proof of this theorem. The first step is to verify that the morphism φ is *$\frac{a}{b}$-power-free* (that is, $\varphi(w)$ is $\frac{a}{b}$-power-free whenever w is $\frac{a}{b}$-power-free). The second step is to verify that φ is *lexicographically least* with respect to $\frac{a}{b}$ (that is, if w is $\frac{a}{b}$-power-free and decrementing any term introduces an $\frac{a}{b}$-power, then decrementing any term in $\varphi(w)$ introduces an $\frac{a}{b}$-power ending at that position). Since the word 0 is $\frac{a}{b}$-power-free and lexicographically least of its length, if φ is an $\frac{a}{b}$-power-free, lexicographically least morphism then $\mathbf{s}_{a/b} = \varphi^{\infty}(0)$. For details, see [3].

We use software to carry out these steps, establishing the structure of several families of sequences $\mathbf{s}_{a/b}$. As a consequence, it follows that these sequences are k-regular for various values of k depending on $\frac{a}{b}$. This suggests the following main question.

Open question. For which rational numbers $\frac{a}{b} > 1$ does there exist an integer k such that $\mathbf{s}_{a/b}$ is k-regular?

References

[1] Jean-Paul Allouche and Jeffrey Shallit, The ring of k-regular sequences, *Theoretical Computer Science* **98** (1992) 163–197.

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[4] Eric Rowland and Jeffrey Shallit, Avoiding $3/2$-powers over the natural numbers, *Discrete Mathematics* **312** (2012) 1282–1288.