Wilf classification of subsets of four-letter patterns

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In the last decades, the problem of avoiding patterns in different combinatorial structures like permutations, coloured permutations, compositions, partitions, set partitions, etc. has been studied by many authors from many different point of views. In this talk, we restrict to permutations and the problem of pattern avoidance for them.

Let $\mathcal{S}_n$ be the symmetric group of all permutations of $[n] \equiv \{1,\ldots,n\}$. Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$ and $\tau = \tau_1\tau_2\cdots\tau_k \in \mathcal{S}_k$ be two permutations. We say that π *contains* τ if there exists a subsequence $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ is *order-isomorphic* to τ, that is, $\pi_{i_a} < \pi_{i_b}$ if and only if $\tau_a < \tau_b$; in such a context τ is usually called a *pattern*. For example, $\pi = 35412$ contains the pattern $\tau = 231$. We say that π *avoids* τ, or is τ*-avoiding*, if such a subsequence does not exist. For example, 35412 avoids 123. The set of all τ-avoiding permutations in $\mathcal{S}_n$ is denoted by $\mathcal{S}_n(\tau)$. For an arbitrary finite collection of patterns T, we say that π *avoids* T if π avoids every pattern τ in T; the corresponding subset of $\mathcal{S}_n$ is denoted by $\mathcal{S}_n(T)$, i.e., $\mathcal{S}_n(T) = \bigcap_{\tau\in T}\mathcal{S}_n(\tau)$. The sets of patterns T and T' belong to the same *Wilf class* (or are *Wilf-equivalent*) if and only if $|\mathcal{S}_n(T)| = |\mathcal{S}_n(T')|$ for all $n \ge 0$.

In 1985, Simion and Schmidt found the cardinality of $\mathcal{S}_n(T)$, where $T \subseteq \mathcal{S}_3$. Thus, the case of patterns of length three is well-known. Let us turn to patterns of length four. For this case, much less is known and is seems hopeless to get an explicit formula for $|\mathcal{S}_n(T)|$ where $T \subseteq \mathcal{S}_4$ is arbitrary. Already the case of avoiding exactly one pattern $\tau \in \mathcal{S}_4$ is not trivial. It was shown that there are three essentially different cases, namely $\mathcal{S}_n(\tau)$ where $\tau \in \{1342, 1234, 1324\}$. Since $|\mathcal{S}_7(1342)| = 2740$, $|\mathcal{S}_7(1234)| = 2761$ and $|\mathcal{S}_7(1324)| = 2762$, these three patterns comprise three different Wilf classes. If we denote the number of symmetry classes and Wilf classes of subsets of k patterns in $\mathcal{S}_4$ by s_k and w_k, respectively, then this means that $w_1 = 3$.

Let us turn to subsets $T = \{\tau_1, \tau_2\}$ with exactly two patterns in $\mathcal{S}_4$. There do exist $\binom{24}{2} = 276$ such subsets T. Le established that these 276 subsets form 38 distinct Wilf classes, i.e., $w_2 = 38$. It seems that the case of k with $3 \le k \le 23$ has not been studied in the literature before the recent work of Mansour and Schork.

Thus, the aim of this talk is discuss how to determine w_k for $3 \leq k \leq 24$. Since the number of subsets of $\mathcal{S}_4$ containing at least 3 patterns is given by $\sum_{k=3}^{24} \binom{24}{k} = 16776915$, it seems to be impossible to reach by constructing explicit bijections between sets of permutations. The way out is to combine several software programs to do the work for us!!

This talk based on recent works of the author with David Callan, Mark Shattuck and Matthias Schork.