Algorithmic Aspects of the Černý Conjecture

Andrzej Kisielewicz

University of Wrocław, Wrocław, Poland, andrzej.kisielewicz@math.uni.wroc.pl

In this talk we present the use of computer search and the role of algorithms in attempts to solve the Černý Conjecture, which is one of the most longstanding open problems in automata theory.

We deal with finite deterministic automata $A = (Q, \Sigma, \delta)$, where Q is a finite set of the states, Σ is a finite input alphabet, and $\delta : Q \times \Sigma \to Q$ is the transition function defining the action of the letters in Σ on Q. The action extends in the natural way to the action of words over Σ on Q and is denoted simply by $qw = \delta(q, w)$.

An automaton A is *synchronizing* if there exist a word w over Σ and a state $q_0 \in Q$ such that for each state $q \in Q$ the image $qw = q_0$. In other words, the word w brings the automaton A to the state q_0 with no regard to in what state it happens to be. Such a word w, if exists, is called a *reset* word for A.

The Černý conjecture states that if an automaton A with n states is synchronizing, then it has a reset word of length not exceeding $(n-1)^2$. It has been proved in many particular cases, but in general, is still open. The best general bound achieved so far for the shortest reset word in synchronizing automata is $(n^3 - n)/6$. The most general result proving the conjecture for a class of automata has been obtained in [2]. (See also [7] for an excellent survey of the topic).

We present our two recent results on the Černý conjecture involving an extensive use of computers and dedicated algorithms. The first concerns the verification of the conjecture for small automata. In [1] all binary automata (that is, those with a two-element alphabet) having at most $n = 9$ states have been checked. Note that there are 9^{18} labeled binary automata with $n = 9$ states, so some more sophisticated approach than *brute force* must be applied. In [1], the authors have managed to restrict the search to the class of the so-called *initially connected* automata. Earlier, the checking of all automata with at most $n = 10$ states was reported in [5], yet no details of computation have been described.

In [4], using a dedicated algorithm for parallel computation, we have verified the conjecture for all binary automata with $n \leq 12$ states. The case of automata with $n = 12$ states took about 100 years of computation time of a single processor core. The number of automata generated by our algorithm in this case was about 10^{15}, which should be compared with about 2.2×10^{17} of non-isomorphic initially connected automata (that one would need to generate applying the technique described in [1]), and 12^{24} of all binary automata with $n = 12$ states.

In [2], the conjecture is considered in terminology of colored digraphs, which refers to the famous Road Coloring problem [6]. We consider edge-colored digraphs with the property that no two edges leaving a vertex have a common color. Such an assignment of colors to edges is called a *road coloring*. Then, given a vertex x, each finite sequence of colors $\alpha_1, \ldots, \alpha_m$ (repetitions allowed) may be considered as a description of a path (road) starting in x and leading to a uniquely determined vertex $y \in V$. (Absence of an edge of a given color α leaving a given vertex x is interpreted as a loop at x colored α).

We are interested in "universal instructions" making it possible to reach a fixed vertex y with no regard at which vertex we start. A sequence of colors $\alpha_1, \ldots, \alpha_m$ such that for each vertex x it describes a path from x to the given y is called a *synchronizing* sequence (for the vertex y).

We define a class of colored digraphs, having a relatively small number of junctions between paths determined by different colors, and prove that the automata corresponding to the digraphs in this class satisfy the Černý conjecture. From computational point of view, we present a number of algorithms finding short synchronizing sequences for various types of graphs. We show that in spite of that the class is defined in a uniform way and the digraphs in the class seem very similar, it requires to apply very different types of algorithms to find a synchronizing sequence short enough.

This suggests that the difficulty in proving the Černý conjecture in its generality may lie in that the solution consists of a large collection of very different algorithmic ideas covering the whole spectrum of synchronizing automata.

References

[1] D. S. Ananichev, M. V. Volkov, and V. V. Gusev. *Primitive digraphs with large exponents and slowly synchronizing automata*, J. Math. Sci. 192, pp. 263–278 (2013).

[2] M. Grech, A Kisielewicz *The Černý conjecture for automata respecting intervals of a directed graph*, Discrete Math. Theoretical Comput. Sci. 15, pp. 61-72 (2013).

[3] M. Grech, A. Kisielewicz *Synchronizing sequences for road colored digraphs*, to appear.

[4] A. Kisielewicz, J. Kowalski, M. Szykuła, *Experiments with Synchronizing Automata*, CIAA 2016, LNCS 9705, pp. 176-188 (2016).

[5] A. N. Trahtman, *An efficient algorithm finds noticeable trends and examples concerning the Černý conjecture*, in Mathematical Foundations of Computer Science, LNCS 4162, pp. 789–800 (2006).

[6] A. N. Trahtman, *The road coloring problem*, Israel J. Math. 172, pp, 51–60 (2009).

[7] M. V. Volkov. *Synchronizing automata and the Černý conjecture*, LATA 2008, LNCS 5196, pp. 11–27 (2008).