Bounds for D-Finite Substitution

Manuel Kauers[1], Gleb Pogudin[1]

[1] *Institute for Algebra, Johannes Kepler University Linz, Austria,*
{manuel.kauers,gleb.pogudin}@jku.at

A function f is called D-finite if it satisfies a linear differential equation with polynomial coefficients,

$$p_0(x)f(x) + p_1(x)f'(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

Typical examples include e^x, $\log(x)$, as well as non-elementary functions such as Bessel functions or the Error function. A function g is called algebraic if it satisfies a polynomial equation,

$$q_0(x) + q_1(x)g(x) + \cdots + p_s(x)g(x)^s = 0.$$

Typical examples include $\sqrt{x}$ or $\sqrt[3]{5x^2-3} + 28x^9$.

It is well-known that every algebraic function is D-finite, and that, more generally, whenever f is D-finite and g is algebraic, then the composition $g \circ f$ is again D-finite. Algorithms for computing a linear differential equation for $g \circ f$ from a given linear differential equation for f and a given polynomial equation for g are part of the standard repertoire of software packages for D-finite functions.

We consider the question how big an equation for $g \circ f$ will be in dependence of the sizes of the equations of f and g. In a first approach, we use a standard argument based on linear algebra: we set up a linear system over the constant field and balance the number of variables and equations. This leads to a so-called order-degree curve, a curve in $\mathbb{R}^2$ such that for all points $(r,d) \in \mathbb{N}^2$ above the curve, there exists an equation for $g \circ f$ of order r with polynomial coefficients of degree at most d.

The order-degree curve obtained in this way is far from tight. In a second approach, we derive a formula for an order-degree curve by analyzing the singularities of the resulting operator. This requires some work because these singularities are not directly accessible from the given data for f and g. However, the work pays off because the resulting curve turns out to be extremely tight, at least generically. We will show some examples during the talk.

Formulas for the resulting curves as well as full details of our derivations can be found in the preprint [1].

References

[1] M. Kauers and G. Pogudin, *Bounds for D-finite Substitution*, ArXiv 1701.07802, Jan 2017.