Bernoulli symbol on multiple zeta values at negative integers

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The multiple zeta functions are defined by, for $\{n_i\}_{i=1}^{r} \subset \mathbb{C}$

$$\zeta_r(n_1,\ldots,n_r) = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1}\cdots k_r^{n_r}}, \tag{1}$$

provided that $\sum_{j=1}^{k} \mathrm{Re}(n_{r+1-j}) > k$, $1 \le k \le r$ ([2, S3]). Values at integer points $\mathbf{n} = (n_1,\ldots,n_r)$ satisfying the constraints are called *multiple zeta values* (MZV). Zhao [4] showed that (1) has an analytic continuation to $\mathbb{C}^r$, not uniquely, due to the *Hartogs' phenomenon*. Thus, several authors have proposed different approaches. For example, Sadaoui [3] used *Raabe's identity* to compute the values.

Theorem 1. *(Sadaoui) [3, eq. (4.10)]*

$$\zeta_r(-n_1,\ldots,-n_r) = \frac{(-1)^r}{n_r+1} \sum_{k_2,\ldots,k_r} \frac{\prod_{j=2}^{r} \mathfrak{A}\left(\sum_{i=j}^{r} (n_i + r - j + 1) - \sum_{i=j+1}^{r} k_i \,\middle|\, k_j \right)}{(\bar{n} + r - \bar{k})} \times \sum_{l_1,\ldots,l_r} \binom{\bar{n}+r-\bar{k}}{l_1} \binom{k_2}{l_2} \cdots \binom{k_r}{l_r} B_{l_1} \cdots B_{l_r},$$

where $k_2,\ldots,k_r \ge 0$, $l_j \le k_j$ *for* $2 \le j \le r$ *and* $l_1 \le \bar{n} + r + \bar{k}$ *with* $\bar{n} = \sum_{j=1}^{r} n_j$, $\bar{k} = \sum_{j=2}^{r} k_j$, $\mathfrak{A}(t|s) := \binom{t}{s}/t$, *and* B_n *is the* n^{th} *Bernoulli number.*

On the other hand, Akiyama and Tanigawa [1] used the Euler-MacLaurin summation formula to obtain results, one of which is the following recurrence. (Here, the notation $\bar{\zeta}$ instead of ζ is used to distinguish two continuations.)

Theorem 2. *(Akiyama and Tanigawa) [1, eq. (15)]*

$$\begin{aligned}\bar{\zeta}_r(-n_1,\ldots,-n_r) = &-\bar{\zeta}_{r-1}(-n_1,\ldots,-n_{r-2},-n_{r-1}-n_r-1)/(n_r+1) \\ &- \bar{\zeta}_{r-1}(-n_1,\ldots,-n_{r-2},-n_{r-1}-n_r)/2 \\ &+ \sum_{q=1}^{n_r} (-n_r)_q\, a_q \bar{\zeta}_{r-1}(-n_1,\ldots,-n_{r-2},-n_{r-1}-n_r+q),\end{aligned}$$

where $(-n_r)_q = (-n_r)(-n_r+1)\cdots(-n_r+q-1)$ *and* $a_q := B_{q+1}/(q+1)!$.

We generalized the idea of Bernoulli symbol to the following $\mathcal{C}$ symbols.

Definition 1. $\mathcal{C}_{1,2,\ldots,k}$ is defined recursively via Bernoulli symbols $\mathcal{B}_1,\ldots,\mathcal{B}_r$ as

$$\mathcal{C}_1^n = \frac{\mathcal{B}_1^n}{n},\ \mathcal{C}_{1,2}^n = \frac{(\mathcal{C}_1+\mathcal{B}_2)^n}{n},\ldots,\ \mathcal{C}_{1,2,\ldots,k+1}^n = \frac{(\mathcal{C}_{1,2,\ldots,k}+\mathcal{B}_{k+1})^n}{n}.$$

Each symbol $\mathcal{C}_{1,2,\ldots,k}$ should be expanded only involving $\mathcal{B}_k$, and then:

1. each power $\mathcal{B}_k^p$ should be evaluated as $\mathcal{B}_k^p \to B_p$;
2. if $k \neq l$, product $\mathcal{B}_k^p\mathcal{B}_l^q$ is evaluated as $\mathcal{B}_k^p\mathcal{B}_l^q \to B_pB_q$.

Theorem 3. *(L. Jiu, V. H. Moll, and C. Vignat)*

$$\zeta_r(-n_1,\ldots,-n_r) = \prod_{k=1}^{r} (-1)^{n_k}\,\mathcal{C}_{1,\ldots,k}^{n_k+1} = \bar{\zeta}_r(-n_1,\ldots,-n_r).$$

Not only do we obtain a symbolic, more compact, effective expression leading to further results such as (denote $\mathbf{a}_k = (a_1,\ldots,a_k)$ for $\mathbf{a} = (a_1,\ldots,a_r)$ and $k<r$)

• recursion formula

$$\zeta_r(-\mathbf{n};\mathbf{z}) = \frac{(-1)^{n_r}}{n_r+1}\sum_{l=0}^{n_r+1}\binom{n_r+1}{l}(-1)^l\,\zeta_{r-1}(-\mathbf{n}_{r-2},-n_{r-1}-l;\mathbf{z})\,B_{n_r+1-l}(z_r);$$

• contiguity identity: for $\mathcal{Z}_r^l = \zeta_r(-\mathbf{n}_{r-1},-n_r-l;\mathbf{z})$;

$$\zeta_r(-\mathbf{n};\mathbf{z}_{r-1},z_r+1) = \zeta_r(-\mathbf{n};\mathbf{z}_{r-1},z_r) + (-1)^{n_r}(z_r-\mathcal{Z}_{r-1})^{n_r};$$

• and generating function

$$\begin{aligned} F_r(w_1,\ldots,w_r) &:= \sum_{n_1,\ldots,n_r\geq 0}\frac{w_1^{n_1}\cdots w_r^{n_r}}{n_1!\cdots n_r!}\zeta_r(-n_1,\ldots,-n_r) \\ &= \big(F_1(w_r,-\partial_{r-1})\cdots F_1(w_2,-\partial_1)\big)\bullet F_1(w_1,0), \end{aligned}$$

where $\partial_i = \partial/\partial w_i$ and $F_1(w,z) = \frac{e^{-wz}}{e^{-w}-1} - \frac{1}{w}$, but also it surprisingly reveals that both analytic continuations in Theorem 1 and Theorem 2 coincide. An explanation of such phenomena is part of future work.

References

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