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The category of finite-dimensional representations of periplectic Lie superalgebras

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Joint work.

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Introduction.

Background: vector superspaces and Lie superalgebras. Work over $\mathbb{C}$.

A $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is defined to be a **vector superspace**.

The **superdimension** of V is defined to be

$$\dim(V) := (\dim V_{\bar{0}} | \dim V_{\bar{1}}) = \dim V_{\bar{0}} - \dim V_{\bar{1}}.$$

Given a homogeneous element $v \in V$, the **parity** (or the **degree**) of v is $\overline{v} \in \{\overline{0}, \overline{1}\}$.
The parity switching functor π sends $V_{\bar{0}} \mapsto V_{\bar{1}}$ and $V_{\bar{1}} \mapsto V_{\bar{0}}$.

Let $m = \dim V_{\bar{0}}$ and $n = \dim V_{\bar{1}}$. Then $\mathfrak{gl}(m|n) := \mathrm{End}_{\mathbb{C}}(V)$ is the associated **Lie superalgebra** of V.

The grading on $\mathfrak{gl}(m|n)$ is induced by V, with **Lie superbracket** (supercommutator) $[x, y] = xy - (-1)^{\overline{x}\overline{y}} yx$ for x, y homogeneous.

Matrix representation for $\mathfrak{gl}(m|n)$.

Given a homogeneous ordered basis for V:

$$V = \underbrace{\mathbb{C}\{v_1, \ldots, v_m\}}_{V_{\overline{0}}} \oplus \underbrace{\mathbb{C}\{v_{1'}, \ldots, v_{n'}\}}_{V_{\overline{1}}},$$

we have

$$\mathfrak{gl}(m|n) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A \in M_{m,m}, B \in M_{m,n}, C \in M_{n,m}, D \in M_{n,n} \right\},$$

where $M_{i,j} := M_{i,j}(\mathbb{C})$. Since $\mathfrak{gl}(m|n) = \mathfrak{gl}(m|n)_{\overline{0}} \oplus \mathfrak{gl}(m|n)_{\overline{1}}$,

$$\mathfrak{gl}(m|n)_{\overline{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\} \text{ and } \mathfrak{gl}(m|n)_{\overline{1}} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\}.$$

We say V is the **natural representation** of $\mathfrak{gl}(m|n)$.

Periplectic Lie superalgebras $\mathfrak{p}(n)$.

Let $m = n$. Then

$$V = \mathbb{C}^{2n} = \underbrace{\mathbb{C}\{v_1, \dots, v_n\}}_{V_{\overline{0}}} \oplus \underbrace{\mathbb{C}\{v_{1'}, \dots, v_{n'}\}}_{V_{\overline{1}}}.$$

Define $\beta : V \otimes V \to \mathbb{C}$ as a nondegenerate (odd) bilinear form satisfying:

$$\beta(v, w) = \beta(w, v), \qquad \beta(v, w) = 0 \quad \text{if } \overline{v} = \overline{w}.$$

We define **periplectic (strange) Lie superalgebras** as:

$$\mathfrak{p}(n) := \{x \in \mathrm{End}_{\mathbb{C}}(V) : \beta(xv, w) + (-1)^{\overline{xv}}\beta(v, xw) = 0\}.$$

In terms of above basis,

$$\mathfrak{p}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{gl}(n|n) : B = B^t, C = -C^t \right\}.$$

The category $\mathcal{F}_n$ of finite-dim'l reps of $\mathfrak{p}(n)$.

The category $\mathcal{F}_n$ of representations of $\mathfrak{p}(n)$:

- objects: finite dimensional $\mathfrak{p}(n)$-representations, integrable over $G_0 = GL_n(\mathbb{C})$,
- morphisms: even G_0-morphisms (so that $\mathcal{F}_n$ becomes abelian); $\mathrm{Hom}_{\mathcal{F}_n}(V, V')$ is a vector space, not a vector superspace.

Note that

- $\mathrm{Hom}_{\mathfrak{p}(n)}(V, V') = \mathrm{Hom}_{\mathcal{F}_n}(V, V') \oplus \mathrm{Hom}_{\mathcal{F}_n}(V, \Pi V')$, and
- $\dim \mathrm{Hom}_{\mathfrak{p}(n)}(V, V') = \dim \mathrm{Hom}_{\mathcal{F}_n}(V, V') + \dim \mathrm{Hom}_{\mathcal{F}_n}(V, \Pi V')$,

where Π is the parity switching functor: $V \cong \Pi V^*$, $V^* \cong V \otimes \Pi\mathbb{C}$, $\Pi\mathbb{C}$ is a 1-dimensional trivial representation in degree $\overline{1}$.

Notations.

$\mathfrak{h}$: standard Cartan of diag matrices in $\mathfrak{g}_0 = \mathrm{Lie}(G_0)$ with dual basis $\{\epsilon_1, \ldots, \epsilon_n\}$.

Weights of $\mathcal{F}_n$ are:

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i=1}^{n} \lambda_i \epsilon_i, \quad \lambda_i \in \mathbb{Z}.$$

We say λ is **(integral) dominant** if and only if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.
Denote Λ_n as the set of dominant weights.
Simple objects (up to $\cong$) of $\mathcal{F}_n$ are parametrized by Λ_n, with the simple module $L(\lambda)$ having the highest weight λ w.r.t. the Borel subalgebra $\mathfrak{b}_0 \oplus \mathfrak{g}_{-1}$.
For $\lambda \in \Lambda_n$, we define

$$|\lambda| = \sum_i \lambda_i, \quad \omega = \sum_{i=1}^{n} \epsilon_i, \quad \rho = \sum_{i=1}^{n} (n-i)\epsilon_i, \quad \overline{\lambda} = \lambda + \rho,$$

$$c_\lambda = \{\overline{\lambda_1}, \ldots, \overline{\lambda_n}\} \subseteq \mathbb{Z}, \quad \gamma = \sum_{\alpha \in \Delta^+(\mathfrak{g}_{-1})} \alpha = \sum_{i=1}^{n} (1-n)\epsilon_i = (1-n)\omega,$$

Notations (continued), Kac modules.

$$\widetilde{\gamma} = \sum_{\alpha \in \Delta^+(\mathfrak{g}_1)} \alpha = \sum_{i=1}^{n} (n+1)\epsilon_i = (n+1)\omega,$$

$$\kappa(\lambda) = \sum_{i \in c_\lambda} (-1)^i, \quad q(\lambda) = \begin{cases} 0 & \text{if } |\lambda| \equiv 0, 1 \mod 4, \\ 1 & \text{if } |\lambda| \equiv 2, 3 \mod 4. \end{cases}$$

Let $V(\lambda)$ be a simple $\mathfrak{g}_0$-module with h.w. λ w.r.t. the Borel $\mathfrak{b}_0 \subseteq \mathfrak{g}_0$.

Let λ be a dominant weight. **Thick Kac module** corresp. to λ is:

$$\Delta(\lambda) = \mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}}^{\mathfrak{g}} V(\lambda) \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0 \oplus \mathfrak{g}_{-1})} V(\lambda),$$

and **thin Kac module** corresp. to λ is:

$$\begin{aligned} \nabla(\lambda) &= \prod^{\frac{n(n-1)}{2}} \mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}} V(\lambda - \gamma) \simeq \mathrm{Coind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}} V(\lambda) \\ &= \mathrm{Hom}_{U(\mathfrak{g}_0 \oplus \mathfrak{g}_1)}(U(\mathfrak{g}), V(\lambda)). \end{aligned}$$

Properties of $\mathcal{F}_n$.

$Z(U(\mathfrak{p}(n)))$ is trivial. So there exists only a few blocks in $\mathcal{F}_n$. Thus adjust the definition of translation functors to obtain finer information.

$U(\mathfrak{p}(n))$ has no quadratic Casimir element. So embed

$$\mathfrak{p}(n) \hookrightarrow \mathfrak{gl}(n|n)$$

and decompose

$$\mathfrak{gl}(n|n) \cong \mathfrak{p}(n) \oplus \mathfrak{p}(n)^{\perp},$$

adjoint and coadjoint representations, $\mathfrak{p}(n)$ and $\mathfrak{p}(n)^{\perp}$, respectively.

$\mathfrak{p}(n)$-invariant element Ω.

Construct a $\mathfrak{p}(n)$-invariant element $\Omega \in \mathfrak{p}(n) \otimes \mathfrak{gl}(n|n)$ as follows:

- $\Omega : M \otimes V \to M \otimes V$, M is a $\mathfrak{p}(n)$-module and $V = \mathbb{C}^{n|n}$ natural rep of $\mathfrak{gl}(n|n)$, where

  $$\Omega = 2 \sum_{x \in \chi} x \otimes x^* = \widetilde{C} - \mathrm{I}_n,$$

  χ is a basis of $\mathfrak{p}(n)$, x^* is dual to x, and $\widetilde{C} \in U(\mathfrak{gl}(n|n))$ is the Casimir element.
- Since actions by Ω and $\mathfrak{p}(n)$ commute on $M \otimes V$, $\Omega \in \mathrm{End}_{\mathfrak{p}(n)}(M \otimes V)$; however, Ω is not an endomorphism of the identity functor for $\mathcal{F}_n$.
- Define $y_\ell : M \otimes V^{\otimes d} \to M \otimes V^{\otimes d}$, $y_\ell = \sum_{i=0}^{\ell-1} \Omega_{i\ell}$, $\Omega_{i\ell}$ is the action on the i-th and ℓ-th factor, and identity otherwise (M module is the 0-th factor).
- $y_1, y_2, \ldots, y_d$ commute pairwise.

Construction of translation functors.

- $\Theta' : \mathcal{F}_n \to \mathcal{F}_n$, where $\Theta' = - \otimes V$.
- Θ' is exact, and $\Theta' \cong \Pi\Theta'$.
- For $k \in \mathbb{C}$, $\Theta'_k : \mathcal{F}_n \to \mathcal{F}_n$ is the functor Θ' followed by projection onto the generalized k-eigenspace for Ω:

$$\Theta'_k(M) = \bigcup_{m>0} \ker(\Omega - kI)^m \Big|_{M \otimes V}.$$

- $\Theta_k := \Pi^k \Theta'_k$ if $k \in \mathbb{Z}$.

Adjunctions

- $\mathrm{Hom}_{\mathcal{F}_n}(\Theta_k M, N) \cong \mathrm{Hom}_{\mathcal{F}_n}(M, \Theta_{k-1} N)$.
- Θ'_k, where $k \in \mathbb{Z}$, are exact.
- For $k \in \mathbb{Z}$, Θ'_k induce $\mathbb{Z}$-linear operators θ'_k on the Grothendieck group G_n of $\mathcal{F}_n$.

Adjunctions, Temperley-Lieb algebra relns.

Corollary

The operators θ'_k on G_n satisfy the relations: for all $k, j \in \mathbb{Z}$, $|k - j| > 1$,

$$\theta'^2_k = 0, \qquad \theta'_k\theta'_j = \theta'_j\theta'_k, \qquad \theta_k\theta'_{k\pm1}\theta'_k = \theta'_k,$$

which are the relations for the Temperley-Lieb algebra on infinite generators.

Theorem

There exists a natural isomorphism of functors: for all $k, j \in \mathbb{Z}$,

$$\Theta^2_k \simeq 0, \qquad \mathrm{adj}_{\Theta_k} = \Psi_{k,k\pm1} : \Theta_k\Theta_{k\pm1}\Theta_k \stackrel{\simeq}{\Rightarrow} \Theta_k$$

$$\Psi_{k,j} = (j - k)1 \otimes s + \mathrm{Id} : \Theta_k\Theta_j \stackrel{\simeq}{\Rightarrow} \Theta_j\Theta_k \qquad \text{if } |k - j| > 1.$$

Θ'_k on Kac modules: λ dominant, $k \in \mathbb{Z}$ fixed.

Lemma (thin Kac module)

There is at most one $1 \leq j \leq n$ such that $\bar{\lambda}_j = k$.

i.) If $\bar{\lambda}_j \neq k$ for all $1 \leq j \leq n$, then $\Theta'_k(\nabla(\lambda)) = 0$.

ii.) If $\bar{\lambda}_j = k$, then $\Theta'_k(\nabla(\lambda))$ can be described by the exact sequence

$$0 \to \nabla(\lambda + \varepsilon_j) \to \Theta'_k(\nabla(\lambda)) \to \Pi\nabla(\lambda - \varepsilon_j) \to 0.$$

Lemma (thick Kac module)

i.) If $\bar{\lambda}_j \neq k, k-2$ for all $j \leq n$, then $\Theta'_k(\Delta(\lambda)) = 0$.

ii.) If $\bar{\lambda}_j = k$ and $\lambda_{j+1} \neq \lambda_j - 1$, then $\Theta'_k(\Delta(\lambda)) = \Delta(\lambda - \varepsilon_j)$.

iii.) If $\bar{\lambda}_j = k - 2$ and $\lambda_{j-1} \neq \lambda_j + 1$, then $\Theta'_k(\Delta(\lambda)) = \Delta(\lambda + \varepsilon_j)$.

iv.) If $\bar{\lambda}_j = k$ and $\lambda_{j+1} = \lambda_j - 1$, then $\Theta'_k(\Delta(\lambda))$ fits into an exact sequence

$$0 \to \Pi\Delta(\lambda - \varepsilon_j) \to \Theta'_k(\Delta(\lambda)) \to \Delta(\lambda + \varepsilon_{j+1}) \to 0.$$

Combinatorics in weight diagrams.

For a dominant weight λ, we define the map

$$f_\lambda : \mathbb{Z} \to \{0, 1\} \quad \text{as} \quad f_\lambda(i) \;=\; \begin{cases} 1 & \text{if } i \in c_\lambda, \\ 0 & \text{if } i \notin c_\lambda, \end{cases}$$

where $c_\lambda := \{\bar{\lambda}_i : i = 1, \dots, n\}$. The corresponding **weight diagram** d_λ is the labeling of the integer line by symbols $\bullet$ ("black ball") and $\circ$ ("empty") such that i has label $\bullet$ if $f(i) = 1$, and label $\circ$ otherwise.

Example: all remaining positions are labeled by $\circ$.

Let $n = 5$. For $\lambda = 0$, the weight diagram d_0 is

…	$\circ$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	…
	-1	0	1	2	3	4	5	6	7	8	9	10	11	

whereas for $\lambda = \rho$, the weight diagram d_ρ is

…	$\circ$	$\bullet$	$\circ$	$\bullet$	$\circ$	$\bullet$	$\circ$	$\bullet$	$\circ$	$\bullet$	$\circ$	$\circ$	$\circ$	…
	-1	0	1	2	3	4	5	6	7	8	9	10	11	

Facts about weight diagrams.

There are two possible partial orders on the weights, corresponding to the choice of either thick Kac modules or thin Kac modules as the standard objects in $\mathcal{F}_n$. In both orders,

$$\text{if } \lambda \leq \mu, \text{ then } \lambda_i \geq \mu_i.$$

In terms of diagrams, this means that the i-th black ball in d_λ (counted from left) lies further to the right of the i-th black ball of d_μ.

This induces a bijection among:

1. the set of dominant weights of $\mathfrak{g} = \mathfrak{p}(n)$,
2. the set of maps $f : \mathbb{Z} \to \{0, 1\}$ such that $\sum_i f(i) = n$, and
3. the set of weight diagrams with exactly n black balls.

Translation of $\Delta(\lambda)$ on weight diagrams.

Translation of $\Delta(\lambda)$ corresp. to moving black balls to position $k-1$.

Proposition (all other positions in the wt diagrams agree)

Let $k \in \mathbb{Z}$. Then

1. $\Theta'_k \Delta(\lambda) \cong \Delta(\mu'')$ if d_λ looks as follows at positions $k-2, k-1, k$

$$d_\lambda : \underset{k-2}{\bullet} \quad \underset{k-1}{\circ} \quad \underset{k}{\circ}$$

$$d_{\mu''} : \underset{k-2}{\circ} \quad \underset{k-1}{\bullet} \quad \underset{k}{\circ}$$

2. $\Theta'_k \Delta(\lambda) = \Pi\Delta(\mu')$ if d_λ looks as follows at positions $k-2, k-1, k$

$$d_\lambda : \underset{k-2}{\circ} \quad \underset{k-1}{\circ} \quad \underset{k}{\bullet}$$

$$d_{\mu'} : \underset{k-2}{\circ} \quad \underset{k-1}{\bullet} \quad \underset{k}{\circ}$$

Translation of $\Delta(\lambda)$ on weight diagrams.

Proposition (continued)

3. If d_λ has positions $k-2, k-1, k$ as below, there exists a SES

$$0 \to \Pi\Delta(\mu') \to \Theta'_k\Delta(\lambda) \to \Delta(\mu'') \to 0$$

where $d_{\mu''}$ and $d_{\mu'}$ are obtained from d_λ by moving one black ball to position $k-1$ (from position $k-2$, respectively, position k):

$$d_\lambda : \quad \underset{k-2}{\bullet} \quad \underset{k-1}{\circ} \quad \underset{k}{\bullet}$$

$$d_{\mu'} : \quad \underset{k-2}{\bullet} \quad \underset{k-1}{\bullet} \quad \underset{k}{\circ}$$

$$d_{\mu''} : \quad \underset{k-2}{\circ} \quad \underset{k-1}{\bullet} \quad \underset{k}{\bullet}$$

4. $\Theta'_k\Delta(\lambda) = 0$ in all other cases.

Translation of $\nabla(\lambda)$ on weight diagrams.

Translation of $\nabla(\lambda)$ corresp. to moving black balls away from k.

Proposition (all other positions in the wt diagrams agree)

Let $k \in \mathbb{Z}$. Then

1. $\Theta'_k \nabla(\lambda) = \Pi\nabla(\mu'')$ if d_λ looks as follows at positions $k-1, k, k+1$

$$\begin{array}{rccc} d_\lambda : & \underset{k-1}{\bullet} & \underset{k}{\bullet} & \underset{k+1}{\circ} \\ d_{\mu''} : & \underset{k-1}{\bullet} & \underset{k}{\circ} & \underset{k+1}{\bullet} \end{array}$$

2. $\Theta'_k \nabla(\lambda) = \nabla(\mu')$ if d_λ looks as follows at positions $k-1, k, k+1$

$$\begin{array}{rccc} d_\lambda : & \underset{k-1}{\circ} & \underset{k}{\bullet} & \underset{k+1}{\bullet} \\ d_{\mu'} : & \underset{k-1}{\bullet} & \underset{k}{\circ} & \underset{k+1}{\bullet} \end{array}$$

Translation of $\nabla(\lambda)$ on weight diagrams.

Proposition (continued)

3. If d_λ looks like below at positions $k-1, k, k+1$, there is a SES

$$0 \to \nabla(\mu') \to \Theta_i' \nabla(\lambda) \to \Pi \nabla(\mu'') \to 0$$

where $d_{\mu'}$ and $d_{\mu''}$ are obtained from d_λ by moving one black ball away from position k (to position $k-1$, respectively, $k+1$):

$$\begin{array}{rccc} d_\lambda: & \underset{k-1}{\circ} & \underset{k}{\bullet} & \underset{k+1}{\circ} \\ d_{\mu'}: & \underset{k-1}{\bullet} & \underset{k}{\circ} & \underset{k+1}{\circ} \\ d_{\mu''}: & \underset{k-1}{\circ} & \underset{k}{\circ} & \underset{k+1}{\bullet} \end{array}$$

4. $\Theta_k' \nabla(\lambda) = 0$ in all other cases.

Arrow diagrams.

For a dominant weight λ define $g_\lambda : \mathbb{Z} \to \{-1, 1\}$ by setting $g_\lambda(i) = (-1)^{f_\lambda(i)+1}$. So $g_\lambda(i) = 1$ if $i \in c_\lambda$, which means d_λ has a black ball at the i-th position and $g_\lambda(i) = -1$ if $i \notin c_\lambda$. For any $j < i$, set

$$r^+(i,j) = \sum_{s=j}^{i-1} g_\lambda(s) \quad \textit{and} \quad r^-(i,j) = -\sum_{s=j+1}^{i} g_\lambda(s).$$

For every $i \in c_\lambda$ define

$$\overset{\longleftarrow i}{\blacktriangle}(\lambda) = \left\{ j < i \mid r^+(i,j) = 0,\ r^+(i,s) \geq 0 \text{ for all } j < s < i \right\}.$$

Also for every $j \notin c_\lambda$ define

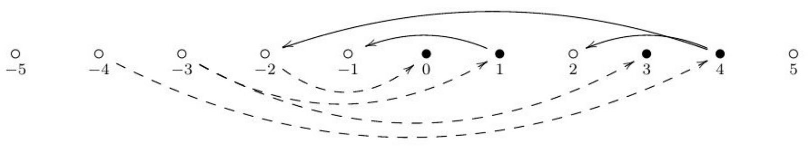
$$\underset{j \dashrightarrow}{\blacktriangledown}(\lambda) = \left\{ i > j \mid r^-(i,j) = 0,\ r^-(s,j) \geq 0 \text{ for all } j < s < i \right\}.$$

Arrow diagrams.

For the *arrow diagram* for λ, equip d_λ with solid and dashed arrows:

- for every $i \in c_\lambda$ draw a solid arrow from i to every $j \in \overset{\leftarrow i}{\blacktriangle}(\lambda)$.
- for every $j \notin c_\lambda$ draw a dashed arrow from j to every $i \in \underset{j\dashrightarrow}{\blacktriangledown}(\lambda)$.

Let $n = 4$, $\lambda = (1, 1, 0, 0)$. Each $i \in c_\lambda$ is marked with a black ball, and for each $i \in c_\lambda$, the positions $j \in \overset{\leftarrow i}{\blacktriangle}(\lambda)$ are connected with i by solid arrows. We also connect by dashed arrows all $j \notin c_\lambda$ with $i \in \underset{j\dashrightarrow}{\blacktriangledown}(\lambda)$:



Note $\overset{\leftarrow 4}{\blacktriangle}(\lambda) = \{-2, 2\}$, $\overset{\leftarrow 0}{\blacktriangle}(\lambda) = \emptyset$, and $\underset{\dashrightarrow -3}{\blacktriangle}(\lambda) = \{1, 3\}$.

Sets of weight diagrams.

We denote by $\blacktriangle(\lambda)$ the set of weight diagrams which are obtained from d_λ by sliding some black balls along solid arrows in the arrow diagram for λ, and by $\blacktriangledown(\lambda)$ the set of weight diagrams obtained by sliding some black balls (backwards) along dashed arrows:

$$\blacktriangle(\lambda) = \left\{ \mu \in \Lambda_n : \forall i \in c_\lambda, f_\mu(i) + \sum_{j \in \overset{\leftarrow i}{\blacktriangle}(\lambda)} f_\mu(j) = 1 \right\}, \tag{1}$$

$$\blacktriangledown(\lambda) = \left\{ \mu \in \Lambda_n : \forall j \notin c_\lambda, 1 - f_\mu(i) + \sum_{i \in \underset{j \dashrightarrow}{\blacktriangledown}(\lambda)} (1 - f_\mu(i)) = 1 \right\}. \tag{2}$$

Proposition

For any dominant weight λ, we have $\blacktriangle(\lambda) \cap \blacktriangledown(\lambda) = \{\lambda\}$.

Equivalence relation on dominant weights.

We define the minimal equivalence relation on the set of dominant weights such that $\lambda \sim \mu$ if μ is obtained from λ by sliding a black ball via a solid or dashed arc.

So $L(\lambda)$ and $L(\mu)$ belong to the same block if and only if $\lambda \sim \mu$.

If we move a black ball via a solid to dashed arrow from position i to position j, then $i \equiv j \mod 2$. Hence since κ is constant on every equivalence class,

$$\kappa(\lambda) = \kappa(\mu) \text{ if and only if } \lambda \sim \mu.$$

Decomposition of $\mathcal{F}_n$ and Θ_i on blocks.

Theorem

The category $\mathcal{F}_n$ has $2(n+1)$ blocks, with

$$\mathcal{F}_n = \bigoplus_{p\in\{-n,-n+2,\ldots,n-2,n\}} (\mathcal{F}_n)_p^+ \oplus \bigoplus_{p\in\{-n,-n+2,\ldots,n-2,n\}} (\mathcal{F}_n)_p^-,$$

where $(\mathcal{F}_n)_p^+$ (resp., $(\mathcal{F}_n)_p^-$) contains all simple modules $L(\lambda)$ with $\kappa(\lambda) = p$ and with parity of h.w. vector equal to $q(\lambda)$ (resp., $q(\lambda)+1$).

Corollary

Let $i \in \mathbb{Z}$, $p \in \{-n, -n+2, \ldots, n-2, n\}$. Then

$$\Theta_i((\mathcal{F}_n)_p^\pm) \subset \begin{cases} (\mathcal{F}_n)_{p+2}^\pm & \text{if } i \text{ is odd and } \frac{n-p}{2} \text{ is even,} \\ (\mathcal{F}_n)_{p+2}^\mp & \text{if } i \text{ is odd and } \frac{n-p}{2} \text{ is odd,} \\ (\mathcal{F}_n)_{p-2}^\pm & \text{if } i \text{ is even and } \frac{n-p}{2} \text{ is even,} \\ (\mathcal{F}_n)_{p-2}^\mp & \text{if } i \text{ is even and } \frac{n-p}{2} \text{ is odd.} \end{cases}$$

Future directions.

Possible directions:

1. Determine the highest weight vectors explicitly in the Kac module in terms of PBW basis and vectors in the Schur functors.
2. Study the branching rules $\mathrm{Res}^{\mathfrak{p}(n)}_{\mathfrak{p}(n-1)} L(\lambda)$ if $L(\lambda)$ is a finite dimensional simple module.
3. Investigate the geometry associated to the enveloping algebra of the periplectic Lie superalgebra.

Thank you.
Questions?