D-finite Numbers*

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D-finite functions have been recognized long ago [6, 5] as an especially attractive class of functions. The defining property of a *D-finite* function is that it satisfies a linear differential equation with polynomial coefficients. In a sense, the theory of D-finite functions generalizes the theory of algebraic functions. Many properties enjoyed by the latter carry over to the former.

It is well-known that the class of algebraic numbers and the class of algebraic functions are naturally connected to each other. Motivated by this relation, we have established in [3] a similar correspondence between numbers and the class of D-finite functions, More precisely, we introduced the following class of numbers.

Definition 1 ([3]). *Let R be a subring of $\mathbb{C}$ and let $\mathbb{F}$ be a subfield of $\mathbb{C}$. A number $\xi \in \mathbb{C}$ is called* D-finite *(with respect to R and $\mathbb{F}$) if there exists a convergent sequence $(a_n)_{n=0}^{\infty}$ over R with $\lim_{n\to\infty} a_n = \xi$ and some polynomials $p_0, \ldots, p_r \in \mathbb{F}[n]$, $p_r \neq 0$, not all zero, such that $p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_r(n)a_{n+r} = 0$ for all $n \in \mathbb{N}$. The set of all D-finite numbers w.r.t. R and $\mathbb{F}$ is denoted by $\mathscr{D}_{R,\mathbb{F}}$.*

It is clear that $\mathscr{D}_{R,\mathbb{F}}$ contains all the elements of R, but it typically contains many further elements. For example, let i be the imaginary unit, then $\mathscr{D}_{\mathbb{Q}(i),\mathbb{Q}(i)}$ contains many (if not all) the periods [4] and, as we will see from Theorem 3, all the values of G-functions [1] as well as many (if not all) regular holonomic constants [2]. In addition, thanks to many mathematicians' work, we can easily recognize that many constants like e, $1/\pi$, Euler's constant γ are D-finite.

The definition of D-finite numbers given above involves two subrings of $\mathbb{C}$ as parameters: the ring to which the sequence terms of the convergent sequences are supposed to belong, and the field to which the coefficients of the polynomials in the recurrence equations should belong. One of the goals of [3] is to investigate how R and $\mathbb{F}$ can be modified without changing the resulting class of D-finite numbers. We have found some interesting properties pursuing this goal. For example, algebraic extensions of $\mathbb{F}$ are useless to extend the class; and it is also not useful to make $\mathbb{F}$ bigger than the quotient field of R. Moreover, we showed that

Theorem 2 ([3]). *For every D-finite number $\xi \in \mathscr{D}_{R,\mathbb{F}}$, there exists $g(z) \in R[[z]]$ D-finite over $\mathbb{F}$ such that $\xi = \lim_{z\to 1^-} g(z)$.*

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The above theorem implies that D-finite numbers are computable when the ring R and the field $\mathbb{F}$ consist of computable numbers. Consequently, all non-computable numbers have no chance to be D-finite. Besides these artificial examples, we do not know of any explicit real numbers which are not in $\mathscr{D}_{\mathbb{Q},\mathbb{Q}}$, and we believe that it may be very difficult to find some.

On the other hand, the values D-finite functions can assume at non-singular algebraic points are all D-finite, as indicated by the following theorem.

Theorem 3 ([3]). *Let $\mathbb{F}$ be a subfield of $\mathbb{C}$ with $\mathbb{F} \setminus \mathbb{R} \neq \emptyset$ and let R be a subring of $\mathbb{C}$ containing $\mathbb{F}$. Assume that $f(z) \in \mathscr{D}_{R,\mathbb{F}}[[z]]$ is analytic at zero and D-finite over $\mathbb{F}$. Further assume that zero and $\zeta \in \bar{\mathbb{F}}$ are not singularities of an annihilating operator for $f(z)$. Then the derivative $f^{(k)}(\zeta) \in \mathscr{D}_{R,\mathbb{F}}$ for all $k \in \mathbb{N}$.*

We have made some first steps in [3] towards understanding the nature of D-finite numbers. We believe that, similarly as for D-finite functions, the class is interesting because it has good mathematical and computational properties and because it contains many special numbers that are of independent interest. At last, we list some possible directions of future research.

1. After proving Theorem 3, it would be natural to wonder about the values of a D-finite function at singularities of its annihilating operators.

2. It would be interesting to know precisely under which circumstances the multiplicative inverse of a D-finite number is D-finite. Are there choices of R and $\mathbb{F}$ for which $\mathscr{D}_{R,\mathbb{F}}$ is a field?

3. A similar pending analogy concerns compositional inverses. Is it true that the values of compositional inverses of D-finite functions are D-finite numbers?

References

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