Computing Automorphism Groups of Designs - a Way to Produce New Symmetric Weighing Matrices

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A weighing matrix of size n and weight k, also denoted as $W(n,k)$ is a $\{0,1,-1\}$-$n\times n$ matrix W such that $WW^T = kI_n$. Two weighing matrices V and W are said to be isomorphic (or Hadamard equivalent), if there exist two signed permutation matrices P and Q auch that $W = PVQ$. In this work we have developed an efficient algorithm, implemented in sage, to find an isomorphism between weighing matrices if one exists. Our algorithm works well with designs in general, and in fact the case of weighing matrices is more difficult because of the presence of signs. In particular, we are able to compute automorphism groups of weighing matrices. One application of this is to search for a (anti-)symmetric weighing matrix in a class of a given matrix W. If a matrix W is isomorphic to W^T, then we compute the isomorphism $PWQ = W^T$, and the automorphism group of W. If a (anti-)symmetric representative of this class exists, then for a specific isomorphism $P'WQ' = W^T$, it will happen that $P'W$ is (anti-)symmetric. We have been able to implement this to a newly discovered weighing matrix $W(23,16)$ and obtain a symmetric matrix with the same parameters.

Our algorithm uses certain strong invariants that may separate nonisomorphic classes. If two matrices V and W have the same invariant, then we have some initial clue on the desired permutations. Then, after considerably small enumeration we are able to reduce the problem to unsigned permutations. Then we use an algorithm based on the singular value decomposition to discover the full permutations.

One interesting (future) application of automorphisms, may apply to the problem of 'coloring' a matrix. Namely, if we are given only the elementwise absolute value $|W|$ of a weighing matrix W, then we need to recover W from $|W|$, at least if we believe that W has rich automorphism group. If we compute the group $Aut(|W|)$, then we need to find a signed permutation group G and an embedding $G \to Aut(|W|)$. If we find such G, then its orbits give us much information as to how to color $|W|$. Finding such G is a problem in Group Theory and it is interesting to understand how the orthogonality of W projects on this Group-Theoretic problem.